Math 7760 – Homework 2 – Due: September 7, 2022

Practice Problems:

Problem 1. Prove that $\dim(\mathrm{Conv}\{v_1, \dots, v_n\}) = \mathrm{rank}(\hat{V}) - 1$ where

$$\hat{V} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ v_1 & v_2 & \dots & v_n \end{pmatrix}$$

Problem 2. Recall from analysis that if $S \subseteq \mathbb{R}^n$ is compact, then each continuous function $f : S \to R$ has a maximum and a minimum on S. Now let $C \subseteq \mathbb{R}^d$ be closed and convex and let $x \in \mathbb{R}^d \setminus C$. Prove that there exists a unique point $y \in C$ minimizing the Euclidean distance to x. [Hint: begin by reducing to the case that C is compact.]

Problems to write up:

Problem 3. Prove the hyperplane separation theorem (see below for the theorem statement and a proof outline).

Theorem (Hyperplane separation theorem)**.** *Given a convex $C \subset \mathbb{R}^n$ and a point $y \in \mathbb{R}^d \setminus C$, there exists $a \in (\mathbb{R}^n)^*$ and $b \in \mathbb{R}$ such that $ax \leq b$ for all $x \in C$ and $ay \geq b$.*

Outline of proof:
Split into the cases based on whether or not $y \in \mathrm{rb}(C)$.
Case 1: $y \notin \mathrm{rb}(C)$:

(1) Prove that there exists a unique z in the closure of C that is nearest to y in the Euclidean distance metric (c.f. Problem 2).
(2) Reduce to the case where $z = -y$.
(3) Show that $y^T x \leq 0$ for all $x \in C$. [Hint: show that if $y^T x > 0$, then $tx + (1-t)z$ and y are closer to each other than z and y are to each other for small positive t.]
(4) Conclude that the hyperplane separation theorem is true when $y \notin \mathrm{rb}(C)$.

Case 2: $y \in \mathrm{rb}(C)$:

(1) Reduce to the case that $y = 0$ and note that it is enough to construct a *linear* hyperplane that does not intersect $\mathrm{relint}(C)$.
(2) Inductively construct a sequence of linear spaces $L_0, \dots, L_{d-1} \subseteq \mathbb{R}^d$ with $\dim L_i = i$, none intersecting $\mathrm{relint}(C)$. [Hint: for $k \leq d-2$, note that $L_k^\perp$ has a two-dimensional subspace P and that $P \cap (C + L_k)$ is convex. Argue that P contains a line L through the origin that does not intersect $P \cap (\mathrm{relint}(C) + L_k)$, and that this implies $L_k + L$ does not intersect $\mathrm{relint}(C)$.]
(3) Conclude that the hyperplane separation theorem is true.