Math 7110 – Homework 4 – Due: Oct 8, 2021

Practice Problems:

Problem 1. Dummit and Foote Section 4.5 problems 4, 19, 26, 30.

Type solutions to the following problems in LaTeX, and email the tex and PDF files to me at `dbernstein1@tulane.edu` by 10am on the due date. Please title them as [lastname].tex and [lastname].pdf. When preparing your solutions, you must follow the rules as laid out in the course syllabus.

Graded Problems:

Problem 2. Solve the two following problems:

(1) Use Sylow's Theorem to prove *Cauchy's theorem*, which says that whenever G is a group whose order is divisible by a prime p, then G contains a subgroup of order p.

(2) Now, use Cauchy's theorem to prove that any *abelian* group of order pq, with p and q prime, is cyclic.

Given a field $\mathbb{F}$, recall that $\mathrm{GL}_n(\mathbb{F})$ is the group (under matrix multiplication) of $n \times n$ nonsingular matrices with entries in $\mathbb{F}$ and that $\mathbb{F}^*$ denotes the group, under multiplication, consisting of all nonzero elements of $\mathbb{F}$ (i.e. $\mathbb{F}^* = \mathrm{GL}_1(\mathbb{F})$). Recall that the map $\det : \mathrm{GL}_n(\mathbb{F}) \to \mathbb{F}^*$ sending a matrix to its determinant is a group homomorphism.

Problem 3. Let $\phi : S_n \to \mathrm{GL}_n(\mathbb{Q})$ be the map sending a permutation σ to the $n \times n$ matrix $\phi(\sigma)$ that has 1 at the $(i, \sigma(i))$ entry for $1 = 1, \dots, n$, and 0 otherwise.

(1) Prove that ϕ is an injective group homomorphism. Matrices of the form $\phi(\sigma)$ are called *permutation matrices.*

(2) Prove that every permutation matrix has determinant ± 1.

(3) Given $\sigma \in S_n$, prove that $\det(\phi(\sigma)) = 1$ if and only if $\sigma \in A_n$.