Math 7110 – Homework 2 – Due: September 22, 2021

Practice Problems:

Problem 1. Section 3.2, questions 14 and 15 (recall that if G acts on S and $s \in S$, then the *stabilizer* of i is the subgroup H of G such that $hs = s$ for all $h \in H$).

Problem 2. Section 3.2, questions 16 and 22.

Type solutions to the following problems in LaTeX, and email the tex and PDF files to me at `dbernstein1@tulane.edu` by 10am on the indicated date. Please title them as [lastname].tex and [lastname].pdf. When preparing your solutions, you must follow the rules as laid out in the course syllabus.

Graded Problems:

Problem 3. The center of a group G is the set $Z(G) = \{z \in G : gz = zg \text{ for all } g \in G\}$.

(1) Show that $Z(G)$ is a normal subgroup of G.
(2) Show that $Z(G) = G$ if and only if G is abelian.
(3) Determine the center of the dihedral group D_n for all n.
(4) Show that $G/Z(G)$ is a cyclic group if and only if G is abelian.

Problem 4. In this problem, you will prove the fundamental group theory result known as *Lagrange's theorem*: if H is a subgroup of G and G is finite, then $|H|$ divides $|G|$.

(1) Let H be a group acting on a set A. Let $\sim$ be the relation on A given by

$$a \sim b \qquad \text{if and only if} \qquad a = hb \qquad \text{for some } h \in H.$$

Prove that $\sim$ is an equivalence relation. The equivalence class of $a \in A$ under $\sim$ is called the *orbit* of a under H.

(2) Now, let H be a subgroup of a finite group G and let H act on G by left multiplication. Let $\mathcal{O}$ be the orbit of some $x \in G$. Prove that the map

$$H \to \mathcal{O} \qquad \text{given by} \qquad h \mapsto hx$$

is a bijection. Now prove that $|H|$ divides $|G|$ and that H has $\frac{|G|}{|H|}$ left cosets in G.