Symmetry-forced rigidity in the plane

Daniel Irving Bernstein

Massachusetts Institute of Technology

dibernst@mit.edu

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Rigidity theory basics

Definition

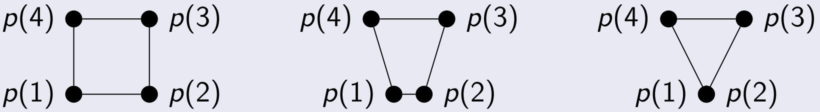
A *bar and joint framework in d dimensions* consists of

- a graph G, and
- a function $p : V(G) \to \mathbb{R}^d$.

Such frameworks can be *rigid* or *flexible*.

Example

Let G be the graph on vertex set $V = \{1, 2, 3, 4\}$ with edges $\{12, 23, 34, 14\}$. Below we give three functions $p : V \to \mathbb{R}^2$. The first two frameworks are flexible and the third one is rigid.



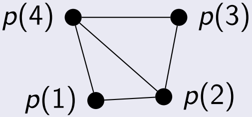
Generic rigidity

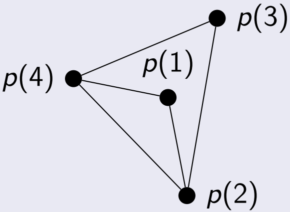
Definition

A graph G is *generically rigid in* $\mathbb{R}^d$ if for every generic $p : V \to \mathbb{R}^d$, the resulting framework is rigid. Such a graph is *minimal* if removing any edge destroys this property.

Example

Let G be the graph on vertex set $\{1, 2, 3, 4\}$ with all edges, aside from $\{1, 3\}$. For generic $p : \{1, 2, 3, 4\} \to \mathbb{R}^2$, the resulting framework is rigid.





Classical results

Question

Which graphs are (minimally) generically rigid in $\mathbb{R}^d$?

Proposition (Folklore)

A graph is generically rigid in $\mathbb{R}^1$ if and only if it is connected.

Theorem (Pollaczek-Geiringer 1927, “Laman’s Theorem”)

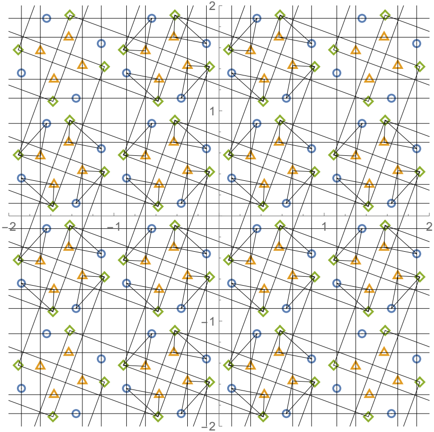
A graph G is minimally generically rigid in $\mathbb{R}^2$ if and only if

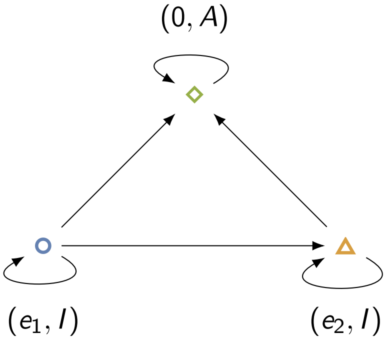
1. $|E(G)| = 2|V(G)| - 3$, *and*
2. $|E(G')| \leq 2|V(G')| - 3$ *for all subgraphs G' of G.*

Generic rigidity in 3 dimensions remains an open problem.

Symmetry-forced rigidity

Frameworks arising in crystallography are infinite and symmetric (Borcea and Streinu 2010). Symmetry-forced rigidity ignores flexes that break the symmetry.





Gain graphs

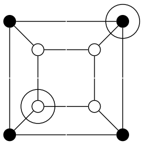
Definition

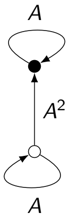
Given a group $\mathcal{S}$, a graph G has *$\mathcal{S}$-symmetry* if there exists a free action of $\mathcal{S}$ on $V(G)$ such that the action of each element of $\mathcal{S}$ is a graph isomorphism of G.

Symmetric frameworks can be compactly represented with *gain graphs*.

Definition

Given a group $\mathcal{S}$, an *$\mathcal{S}$-gain graph* is a directed multigraph G whose arcs are labeled by elements of $\mathcal{S}$.





A is a rotation 90° counterclockwise.

$$\mathcal{S} = \{I, A, A^2, A^3\}$$

Balanced cycles

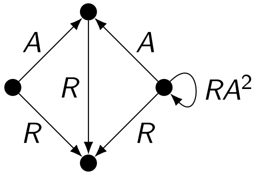
Definition

The *gain* of a walk W in a gain graph G is the product of the labels in W, inverting when an arc is traversed backwards. A *balanced cycle* is a cycle whose gain is the identity.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{S} = \{I, A, A^2, A^3, R, RA, RA^2, RA^3\}$$



Balancedness of a cycle not affected by:

- reversing an edge and inverting its label
- starting at a different vertex

Generic symmetry-forced rigidity

Theorem (Ross 2014, Malstein and Theran 2013)

Let $\mathcal{S}$ be a lattice of plane translations. An $\mathcal{S}$-gain graph G is minimally generically rigid in $\mathbb{R}^2$ iff G has $2|V(G)|-2$ edges and every sub-gain-graph G' of G satisfies

$$|E(G')| \leq \begin{cases} 2|V(G')|-3 & \text{if every cycle in } G' \text{ is balanced} \\ 2|V(G')|-2 & \text{otherwise.} \end{cases}$$

Theorem (Jordán, Kaszanitzky, and Tanigawa 2016, Malstein and Theran 2015)

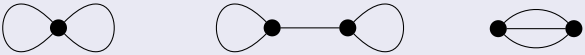
Let $\mathcal{S}$ be a finite group of plane rotations and let G be an $\mathcal{S}$-gain graph. Then G is minimally generically rigid in $\mathbb{R}^2$ iff G has $2|V(G)|-1$ edges and every sub-gain-graph G' of G satisfies

$$|E(G')| \leq \begin{cases} 2|V(G')|-3 & \text{if every cycle in } G' \text{ is balanced} \\ 2|V(G')|-1 & \text{otherwise.} \end{cases}$$

Dutch bicycles and complete gain graphs

Definition

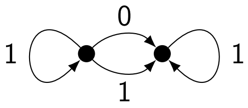
A *bicyclic graph* is a subdivision of one of the following graphs:



A bicyclic gain graph is *Dutch* if each pair of closed walks based at the same vertex have gains that commute.

Definition

Given a group $\mathcal{S}$, the *complete gain graph* $K_n(\mathcal{S})$ has vertex set $\{1, \ldots, n\}$ and $|\mathcal{S}|$ arcs from i to j when $i < j$ and $|\mathcal{S}| - 1$ loops at each vertex. Each non-loop edge between i and j is labeled by a distinct element of $\mathcal{S}$ and each loop edge is labeled by a distinct non-identity element of $\mathcal{S}$.

$K_2(\mathbb{Z}_2) =$ 

The main theorem

Theorem (B. 2020)

Let $\mathcal{S}$ be a subgroup of $\mathbb{R}^2 \rtimes SO(2)$. For each $\mathcal{S}$-gain graph H, define

$$\alpha(H) = \begin{cases} 3 & \text{if every cycle in } H \text{ is balanced} \\ 2 & \text{if not, and the gain of each cycle is a translation} \\ 1 & \text{if none of the above, and all bicyclic subgraphs are Dutch} \\ 0 & \text{otherwise.} \end{cases}$$

Then G is minimally generically infinitesimally rigid in $\mathbb{R}^2$ if and only if

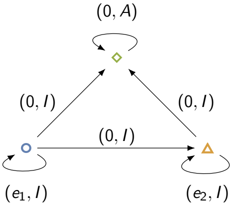
$$|E(G)| = 2|V(G)| - \alpha(K_{|V(G)|}(\mathcal{S}))$$

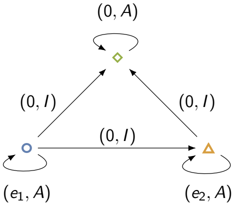
and for all subgraphs G' of G,

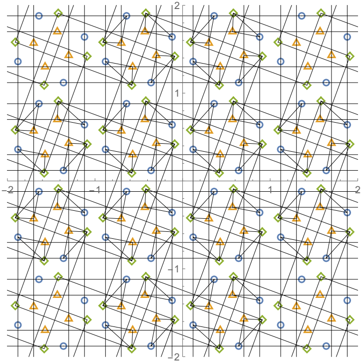
$$|E(G')| \leq 2|V(G')| - \alpha(G').$$

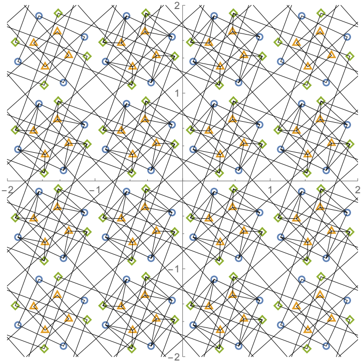
See also work of Malestein and Theran for generic symmetry-forced rigidity for wallpaper groups with variable representations

Example









Recall that composition in $\mathbb{R}^2 \rtimes SO(2)$ is given by $(b_1, A_1)(b_2, A_2) = (b_1 + A_1 b_2, A_1 A_2)$.

Outline of proof

- $\mathcal{S}$-symmetry forced rigid graphs are spanning sets in algebraic matroid of $\mathcal{S}$-*symmetric Cayley-Menger variety*
- When $\mathcal{S} \subseteq \mathbb{R}^2 \rtimes SO(2)$, this is a Hadamard product of affine spaces
- Each affine space defines two matroids, one which is an elementary lift of the other
- Describe the algebraic matroid of a Hadamard product of affine spaces in terms of these two matroids for each (proof uses tropical geometry)
- Apply to our setting - involves a particular lift of the gain graphic matroid of a complete gain graph

Algebraic matroids

Each subset $S \subseteq E$ defines a coordinate projection $\pi_S : \mathbb{C}^E \to \mathbb{C}^S$.

Definition

Let $V \subseteq \mathbb{C}^E$ be an irreducible variety. A given $S \subseteq E$ is

1. *independent* if $\dim(\pi_S(V)) = |S|$,
2. *spanning* if $\dim(\pi_S(V)) = \dim(V)$, and
3. *a basis* if S is both independent and spanning.

The common combinatorial structure described by any one of these set systems is called the *algebraic matroid underlying* V.

Let $E = \{1,2,3\} \times \{1,2,3\}$ and $V \subseteq \mathbb{C}^E$ be the variety of 3×3 matrices of rank ≤ 1. Then $S := \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,3)\}$ is spanning, but not independent.

$$\pi_S(V) = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & \cdot \\ \cdot & \cdot & x_{33} \end{pmatrix} : x_{11}x_{22} - x_{21}x_{22} = 0 \right\}$$

Algebraic matroids in rigidity theory

Definition

Given a pair of integers $d \leq n$, the *Cayley-Menger variety of n points in* $\mathbb{R}^d$, denoted CM_n^d, is the affine variety embedded in $\mathbb{C}^{\binom{[n]}{2}}$ as the Zariski closure of the set of possible squared pairwise euclidean distances between n points in $\mathbb{R}^d$.

Example

Let $d = 2$. Then the ij coordinate of CM_n^2 is parameterized as $d_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2$.

Observation

A graph $G = ([n], E)$ is generically rigid in $\mathbb{R}^d$ if and only if E is spanning in CM_n^d. Moreover, G is minimally generically rigid if and only if E is a basis of CM_n^d.

Matroids

Definition

A *matroid* is a pair $\mathcal{M} = (E, \mathcal{I})$ where E is a set and $\mathcal{I} \subseteq 2^E$ satisfies

1. $\mathcal{I}$ is nonempty,
2. if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$, and
3. if $I, J \in \mathcal{I}$ with $|I| = |J| + 1$, then there exists $e \in I$ such that $J \cup \{e\} \in \mathcal{I}$.

Elements of $\mathcal{I}$ are called the *independent sets of* $\mathcal{M}$.

Definition

The *rank function* $r_{\mathcal{M}} : 2^E \to \mathbb{Z}_{\geq 0}$ of a matroid $\mathcal{M} = (E, \mathcal{I})$ maps $S \subseteq E$ to $|I|$ where I is the largest independent subset of S.

Definition

A *spanning set* of $\mathcal{M} = (E, \mathcal{I})$ is a set $S \subseteq E$ of maximum rank. A *basis* is a spanning independent set.

Matroids from submodular functions

Definition (Edmonds and Rota 1966)

Let $f : 2^E \to \mathbb{Z}$ be increasing and submodular, i.e. satisfies

1. $f(A) \leq f(B)$ whenever $A \subseteq B \subseteq E$
2. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$.

Define $\mathcal{M}(f)$ to be the matroid on E where $I \subseteq E$ is independent iff

$$\text{for all } I' \subseteq I, \quad I' = \emptyset \quad \mathit{or} \quad |I'| \leq f(I').$$

Example (Pym and Perfect 1970)

If $r_1, \dots, r_d$ are rank functions of matroids $M_1, \dots, M_d$ on ground set E, then I is independent in $\mathcal{M}(r_1 + \cdots + r_d)$ iff $I = I_1 \cup \cdots \cup I_d$ where I_j is independent in M_j.

Hadamard product of varieties

Definition

The *Hadamard* product $u \star v$ of $u, v \in \mathbb{F}^E$ is $(u_e v_e)_{e \in E}$. The *Hadamard product* of varieties U, V is the Zariski closure of $\{u \star v : u \in U, v \in V\}$.

Theorem (B. 2020)

Let $U, V \subseteq \mathbb{C}^E$ be linear spaces. Then $\mathcal{M}(U \star V) = \mathcal{M}(r_{\mathcal{M}(U)} + r_{\mathcal{M}(V)} - 1)$.

Proposition

$CM_n^2 = U \star U$ where U is the linear space spanned by the incidence matrix of the complete graph on n vertices.

Corollary (Lovász and Yemini 1982)

Let r be the rank function of the graphic matroid underlying K_n. Then $\mathcal{M}(2r - 1)$ is the algebraic matroid underlying CM_n^2.

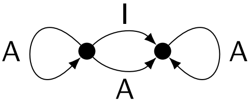
Symmetric Cayley-Menger varieties

- $\mathcal{S}$ is a group of Euclidean isometries of $\mathbb{R}^d$
- $\mathbb{F}^{K_n(\mathcal{S})}$ denotes the $\mathbb{F}$-vector space with coordinates indexed by the arcs of $K_n(\mathcal{S})$
- Define $d : (\mathbb{R}^d)^n \to \mathbb{R}^{K_n(\mathcal{S})}$ by

$$d(z)_e := \|z_{\text{source}(e)} - \text{gain}(e) z_{\text{target}(e)}\|_2^2.$$

- $CM_n^{\mathcal{S}}$ is the Zariski closure of the image of d
- $\mathcal{S}$-gain graph G is generically infinitesimally rigid iff spanning in $CM_n^{\mathcal{S}}$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathcal{S} = \{A, I\}$$



$$d(x_1, y_1, x_2, y_2) = \Big(4x_1^2 + 4y_1^2, \quad (x_1 - x_2)^2 + (y_1 - y_2)^2, \\ (x_1 + x_2)^2 + (y_1 + y_2)^2, \quad 4x_2^2 + 4y_2^2\Big)$$

Translations and rotations

$d = 2$ and $\mathcal{S}$ is a subgroup of $\mathbb{R}^2 \rtimes SO(2)$. If arc e of $K_n(\mathcal{S})$ has gain

$$\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\right)$$

then under the following change of parameters

$$x_v \mapsto \frac{x_v + y_v}{2} \qquad y_v \mapsto \frac{x_v - y_v}{2i}$$

the entry of $CM_n^{\mathcal{S}}$ corresponding to e is

$$(x_{\mathrm{source}(e)} - e^{i\theta} x_{\mathrm{target}(e)} - a - bi)(y_{\mathrm{source}(e)} - e^{-i\theta} y_{\mathrm{target}(e)} - a + bi)$$

and so $CM_n^{\mathcal{S}}$ is a Hadamard product of affine spaces.

Algebraic matroid of a Hadamard product of affine spaces

Let $V = \{Ax + b : x \in \mathbb{C}^d\} \subseteq \mathbb{C}^E$ be an affine space.

- the algebraic matroid $\mathcal{M}(V)$ of V is the row matroid of A
- define $\mathcal{M}^L(V)$ to be the row matroid of $(A\ b)$
- $\mathcal{M}^L(V)$ is an elementary lift of $\mathcal{M}(V)$
- $I \subseteq E$ is independent in $\mathcal{M}(V)$ implies I independent in $\mathcal{M}^L(V)$

Theorem (B. 2020)

Let $U, V \subseteq \mathbb{C}^E$ be finite-dimensional affine spaces and define $f : 2^E \to \mathbb{Z}$ by

$$f(S) = \begin{cases} r_{\mathcal{M}(U)}(S) + r_{\mathcal{M}(V)}(S) & \text{if } r_{\mathcal{M}(U)}(S) < r_{\mathcal{M}^L(U)}(S) \\ & \text{ or } r_{\mathcal{M}(V)}(S) < r_{\mathcal{M}^L(V)}(S) \\ r_{\mathcal{M}(U)}(S) + r_{\mathcal{M}(V)}(S) - 1 & \text{otherwise.} \end{cases}$$

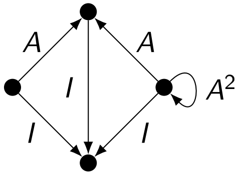
Then $\mathcal{M}(U \star V) = \mathcal{M}(f)$.

Gain graphic matroids

Definition

The *gain-graphic matroid* of a gain graph G is the matroid supported on the arc set of G whose independent sets are sets of arcs such that each connected component has at most one cycle, which is not balanced.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \mathcal{S} = \{I, A, A^2, A^3\}$$



Example

In the above gain graph, adding the loop to any spanning tree produces a basis of the underlying gain-graphic matroid.

Putting it all together

If $\mathcal{S} \subseteq \mathbb{R}^2 \rtimes SO(2)$, $CM_n^{\mathcal{S}} = U \star V$ where U, V are affine spaces satisfying

- $\mathcal{M}(U) = \mathcal{M}(V)$ is the gain-graphic matroid of the gain graph obtained from $K_n(\mathcal{S})$ by ignoring the translation part of each gain
- $\mathcal{M}^L(U) = \mathcal{M}^L(V)$ is obtained from the gain graph of $K_n(\mathcal{S})$ by making non-Dutch bicyclic subgraphs independent

Theorem (B. 2020)

Let $\mathcal{S}$ be a subgroup of $\mathbb{R}^2 \rtimes SO(2)$. For each $\mathcal{S}$-gain graph H, define

$$\alpha(H) = \begin{cases} 3 & \text{if every cycle in } H \text{ is balanced} \\ 2 & \text{if not, and the gain of each cycle is a translation} \\ 1 & \text{if none of the above, and all bicyclic subgraphs are Dutch} \\ 0 & \text{otherwise.} \end{cases}$$

Then G is independent in $\mathcal{M}(CM_n^{\mathcal{S}})$ if and only if $|E(G')| \leq 2|V(G')| - \alpha(G')$ for all subgraphs G' of G.

The end

Thank you for your attention!

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