ALGEBRAIC MATRIX COMPLETION PROBLEMS

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This documents contains several open problems (as of the time of writing) related to the algebraic geometry of low-rank matrix completion problems. They are to be discussed (and hopefully solved!) at the ICERM working group on matrix completion. For more background, consult the references listed at the end. A list of the working group participants can be found here:

`https://docs.google.com/spreadsheets/d/1_2RAPcDqnVM1_2YxYW6z-MnqUbpiPMbWXGYWaOQUCbo/edit#gid=0`

1. Typical ranks of non-symmetric partial matrices

Problem 1. SOLVED ~~What is the maximum typical rank of the graph corresponding to non-symmetric partial matrices of the following form? Question marks correspond to unknown entries and stars correspond to known entries.~~

$$\begin{pmatrix} ? & * & * & * & * \\ * & ? & * & * & * \\ * & * & ? & * & * \\ * & * & * & ? & * \\ * & * & * & * & ? \end{pmatrix}.$$

~~What about the case of arbitrary $n \times n$ partial matrices with unknown diagonal?~~

Problem 2. Let G be a planar bipartite graph. If G is not a tree, then G has generic completion rank 2. Therefore the greatest possible typical rank that G can have is 3. Under what conditions on G is 3 obtained as a typical rank? More generally, when does a given graph of generic completion rank 2 also have 3 as a typical rank? One necessary condition here is that G have a non-empty 2-core. Is this necessary condition sufficient?

Problem 3. Develop (numerical) techniques for computing the typical ranks of a bipartite graph.

Problem 4. Find classes of graphs whose generic completion rank is predicted by a dimension count.

2. Typical ranks of symmetric partial matrices

Problem 5. SOLVED ~~Characterize the semi-simple graphs on n vertices that have n as a typical rank.~~

Problem 6. Develop (numerical) techniques for computing the typical ranks of a semisimple graph.

Problem 7. Find classes of graphs whose generic completion rank is predicted by a dimension count.

3. Generic completion rank 2, non-symmetric case

Problem 8. Let $G = ([m], [n], E)$ be a bipartite graph and consider the projection $\pi_E : \mathrm{Mat}_2^{m\times n} \to \mathbb{C}^E$ of the variety of $m \times n$ matrices of rank at most 2 onto the coordinates indexed by E. When is the corresponding elimination ideal generated by 3×3 minors? *Problem suggested by Elizabeth Gross.*

Before attempting the next two problems, one might wish to familiarize themself with the characterization of the independent sets in the algebraic matroid underlying the variety of $m \times n$ matrices of rank at most two given in [1].

Problem 9. Does there exist a polynomial-time algorithm to check that a given bipartite graph has generic completion rank 2? Exhibit such an algorithm, or show that this problem is NP-hard.

Problem 10. Given a basis of the algebraic matroid underlying the variety of $m\times n$ matrices of rank at most 2, what is the degree of the corresponding projection map?

Problem 11. Find a nice characterization of the circuits of the algebraic matroid underlying the variety of $m \times n$ matrices of rank at most 2.

Following a solution to Problem 11, one might attempt the following.

Problem 12. Find a combinatorial algorithm that takes one of these circuits as input, and outputs the corresponding circuit polynomial.

4. Generic completion rank 2, symmetric case

Unlike in the non-symmetric (and skew-symmetric) case, there is no known combinatorial characterization of the algebraic matroid underlying the variety of $n \times n$ symmetric matrices of rank at most 2. Hence the only problem listed here is quite broad.

Problem 13. Find a combinatorial description of the algebraic matroid underlying the variety of symmetric matrices of rank at most 2.

5. Maximum likelihood threshold

Problem 14. For which planar graphs is the maximum likelihood threshold four?

Problem 15. Develop (numerical) techniques for computing maximum likelihood thresholds.

References

[1] Daniel Irving Bernstein. Completion of tree metrics and rank 2 matrices. *Linear Algebra and its Applications*, 533:1–13, 2017. arXiv:1612.06797.

[2] Daniel Irving Bernstein, Grigoriy Blekherman, and Rainer Sinn. Typical and generic ranks in matrix completion. *arXiv preprint*, 2018. arXiv:1802.09513.

[3] Grigoriy Blekherman and Rainer Sinn. Maximum likelihood threshold and generic completion rank of graphs. *Discrete & Computational Geometry*, pages 1–22, 2017. arXiv:1703.07849.

[4] Elizabeth Gross and Seth Sullivant. The maximum likelihood threshold of a graph. *Bernoulli*, 24(1):386–407, 2018. arXiv:1404.6989.