Tropical Geometry for Rigidity Theory and Matrix Completion

Daniel Irving Bernstein

North Carolina State University

dibernst@ncsu.edu
http://www4.ncsu.edu/~dibernst/

Algebraic Matroids

Definition (Algebraic matroid)

Let S be a finite set and let $V \subseteq \mathbb{K}^S$ be an irreducible variety. Let $E \subseteq S$. Denote by $\pi_E : \mathbb{K}^S \to \mathbb{K}^E$ the corresponding projection map. Then E is:

1. *independent* if $\pi_E(V) \subseteq \mathbb{K}^E$ is full dimensional
2. *spanning* if $\dim(\pi_E(V)) = \dim(V)$

The set of independent/spanning sets is called the *algebraic matroid of V*

Let $S = \{1,2,3\} \times \{1,2,3\}$ so $\mathbb{C}^S$ is the set of 3×3 matrices
Let $V \subseteq \mathbb{C}^S$ be the set of 3×3 matrices of rank 1
Define $E := \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$.
E is neither independent nor spanning since the dimension of

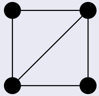
$$\pi_E(V) = \left\{ \begin{pmatrix} x_{11} & x_{12} & \cdot \\ x_{21} & x_{22} & \cdot \\ \cdot & \cdot & x_{33} \end{pmatrix} : x_{11}x_{22} - x_{21}x_{22} = 0 \right\}$$

is 4 whereas $|E| = 5$ and $\dim(V) = 5$.

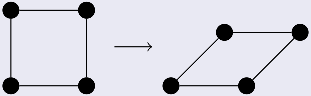
Algebraic matroids in Rigidity Theory

Question

Given an embedding of a graph $G = (E, V)$ inside $\mathbb{R}^d$ is the resulting structure rigid, even allowing the edges to move freely about the vertices?



(a) A rigid framework in $\mathbb{R}^2$



(b) A flexible framework in $\mathbb{R}^2$

For $i = 1 \dots n$, define $\mathbf{t}_i = (t_i^1, \dots, t_i^d)$ to be a d-vector of variables. The *Caley-Menger variety* $CM_n^d \subseteq \mathbb{R}^{\binom{[n]}{2}}$ is parameterized by $x_{ij} = \|\mathbf{t}_i - \mathbf{t}_j\|_2^2$.

Proposition (Folklore?)

A generic embedding of graph $G = ([n], E)$ into $\mathbb{R}^d$ is rigid if and only if E is spanning in the algebraic matroid underlying CM_n^d.

Algebraic Matroids in low-rank matrix completion

Let $\mathcal{M}_r^{m\times n} \subseteq \mathbb{C}^{[m]\times[n]}$ be the variety consisting of all $m \times n$ matrices of rank at most r.

Proposition

$E \subseteq [m] \times [n]$ is spanning in the algebraic matroid underlying $\mathcal{M}_r^{m\times n}$ if and only if $\pi_E^{-1}(\pi_E(M))$ is finite for generic $M \in \mathcal{M}_r^{m\times n}$.

Example

$E = \{11, 12, 21\}$ is spanning in the algebraic matroid underlying $\mathcal{M}_1^{2\times 2}$ but $F = \{11, 12\}$ is not.

$$\pi_E(M) = \begin{pmatrix} 1 & 2 \\ 3 & \cdot \end{pmatrix} \qquad \pi_F(N) = \begin{pmatrix} 4 & 5 \\ \cdot & \cdot \end{pmatrix}$$

From $\pi_E(M)$, we know that the unobserved entry of M is 6. There are infinitely many possibilities for the missing entries of $\pi_F(N)$.

General setup

Problem

Let S be a finite set and $V \subseteq \mathbb{C}^S$ be a variety. Describe combinatorially the subsets $E \subseteq S$ that are independent in the algebraic matroid underlying V.

When $S = \binom{[n]}{2}$, we associate $E \subseteq S$ with graph $([n], E)$.
When $S = [m] \times [n]$, we associate $E \subseteq S$ with bipartite graph $([m], [n], E)$.
Characterizations already known for:

1. $V = \mathcal{M}_1^{m \times n}$ ($S = [m] \times [n]$) (Folklore)
2. $V = CM_n^1$ ($S = \binom{[n]}{2}$) (Folklore)
3. $V = CM_n^2$ ($S = \binom{[n]}{2}$) (Laman, 1970)

Using tropical geometry, we give characterizations for:

1. $V = \mathcal{M}_2^{m \times n}$ ($S = [m] \times [n]$)
2. $V = \mathcal{S}_2^n$, the variety of $n \times n$ skew-symmetric rank ≤ 2 matrices ($S = \binom{[n]}{2}$)

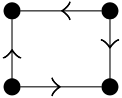
Characterization of algebraic matroid of $\mathcal{M}_r^{m \times n}$

Theorem (Folklore)

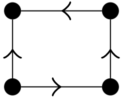
The set E is independent in the algebraic matroid underlying $\mathcal{M}_1^{m\times n}$ if and only if the bipartite graph $([m],[n],E)$ has no cycles.

Theorem (B-, 2016)

A subset $E \subseteq [m] \times [n]$ is independent in the algebraic matroid underlying $\mathcal{M}_2^{m\times n}$ if and only if there exists some acyclic orientation of $([m],[n],E)$ that has no alternating cycles.



Alternating cycle



Non-alternating cycle

Skew-symmetric determinantal variety

Denote by $\mathcal{S}_2^n$ the variety of $n \times n$ skew-symmetric matrices of rank ≤ 2.

Theorem (B-, 2016)

A set $E \subseteq \binom{[n]}{2}$ is independent in the algebraic matroid underlying $\mathcal{S}_2^n$ if and only if there exists an acyclic orientation of the the graph $([n], E)$ that has no alternating closed trail.

If $k = m + n$ then $\mathcal{M}_2^{m\times n}$ is a projection of $\mathcal{S}_2^k$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \to \quad \begin{pmatrix} 0 & e & \mathbf{a} & \mathbf{b} \\ -e & 0 & \mathbf{c} & \mathbf{d} \\ -\mathbf{a} & -\mathbf{b} & 0 & f \\ -\mathbf{c} & -\mathbf{d} & -f & 0 \end{pmatrix}$$

Therefore, result for $\mathcal{M}_2^{m\times n}$ follows immediately from result for $\mathcal{S}_2^k$

Tropical geometry

Associated to a complex variety $V \subseteq \mathbb{C}^n$ is a polyhedral fan $\mathrm{trop}(V) \subseteq \mathbb{R}^n$ known as the *tropicalization of V*.

Theorem (Bieri, Groves, Bogart, Jensen, Speyer, Sturmfels, Thomas)

The tropicalization of an irreducible complex variety is the support of a pure balanced polyhedral fan that is connected through codimension 1.

Proposition (Yu 2016)

Given any irreducible variety $V \subseteq \mathbb{C}^S$, a subset $E \subseteq S$ is independent in the corresponding algebraic matroid if and only if the projection of $\mathrm{trop}(V) \subseteq \mathbb{R}^S$ onto $\mathbb{R}^E$ is all of $\mathbb{R}^E$.

Theorem (Speyer-Sturmfels 2003)

The tropicalization of the set of $n \times n$ rank-2 skew symmetric matrices, $\mathrm{trop}(\mathcal{S}_2^n)$, is the set of tree metrics on a set of size n.

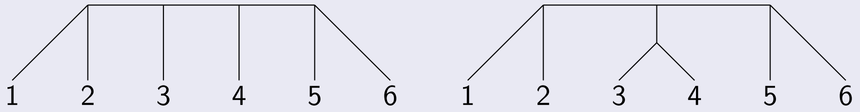
The upshot and a key lemma

Proposition

1. $\mathrm{trop}(\mathcal{S}_2^n)$ *is a polyhedral fan*
2. *There is a bijection between its maximal cones and binary trees on n labeled leaves (Notation: each tree T gives cone K_T)*
3. $E \subseteq \binom{[n]}{2}$ *is independent in $\mathcal{S}_2^n$ iff there exists some maximal cone K_T of* $\mathrm{trop}(\mathcal{S}_2^n)$ *such that E is independent* $\mathrm{span}_{\mathbb{R}}(K_T)$

Key Lemma

A set $E \subseteq \binom{[n]}{2}$ is independent in the matroid of $\mathrm{span}(K_T)$ *for some tree T iff E is independent in the matroid of* $\mathrm{span}(K_C)$ *for some caterpillar C.*

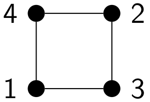


(a) caterpillar (b) not a caterpillar

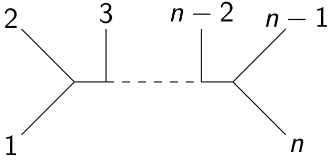
Caterpillar matroids

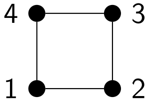
Proposition

Let C be the caterpillar whose leaves are labeled $1, \ldots, n$ from left to right. Then $E \subseteq \binom{[n]}{2}$ gives an independent set of $\mathrm{span}_{\mathbb{R}} K_C$ if and only if the graph $([n], E)$ has no closed walks with alternating vertices.



Dependent Set





Independent Set

Theorem (B-. 2016)

A set $E \subseteq \binom{[n]}{2}$ is independent in the algebraic matroid underlying $\mathcal{S}_2^n$ if and only if there exists a permutation σ of $[n]$ such that the graph $([n], \sigma E)$ has no closed trail with alternating vertices.

Open problems

- Can an independent set of $\mathcal{M}_2^{m\times n}$ or $\mathcal{S}_2^n$ be recognized in polynomial time?
- Find a “nice” combinatorial description of the circuits of the algebraic matroids underlying $\mathcal{M}_2^{m\times n}$ and $\mathcal{S}_2^n$
- Develop a combinatorial algorithm to compute a generating set of the elimination ideals corresponding to coordinate projections of $\mathcal{M}_2^{m\times n}$ and $\mathcal{S}_2^n$
- Find a constructive classification of the bases $\mathcal{M}_2^{m\times n}$ and $\mathcal{S}_2^n$ analogous to the Henneberg moves for Laman graphs
- What other algebraic matroids can be classified using tropical methods?
- Find a more basic proof of our main results

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