Schur Complement and Matrix Completion

Fei Wang
f49wang@uwaterloo.ca

University of Waterloo

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Background

Suppose that we are given a rank r matrix M of size $m \times n$ where a subset of entries are sampled.

The goal is to recover the missing entries from the known entries of matrix M.

$$\begin{array}{ll} \textit{Find} & M \\ \text{s.t.} & \mathcal{P}_{\hat{E}}(M) = z \\ & \operatorname{rank}(M) = r \end{array} \tag{1}$$

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We will discuss the sufficient and necessary conditions such that a completion of the matrix is unique. Our main tool is Schur Complement.

Recall the basic property of Schur complement:

Lemma 1

[1] Consider the partitioned matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{m \times n},$$

if we assume that $\mathrm{Range}(B) \subseteq \mathrm{Range}(A)$ *and* $\mathrm{Range}(C^T) \subseteq \mathrm{Range}(A^T)$, *then* $M/A = D - CA^\dagger B$ *is well-defined and*

$$\begin{bmatrix} I & 0 \\ -CA^\dagger & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -A^\dagger B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & M/A \end{bmatrix}$$

and hence

$$\mathrm{rank}(M) = \mathrm{rank}(A) + \mathrm{rank}(M/A).$$

Here we can assume $A^\dagger$ is the Moore-Penrose pseudo inverse.

We first start with a simple case.

Theorem 2

Consider the partitioned matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{m \times n},$$

where $\operatorname{rank}(M) = r$*, and the blocks* A, B, C *are fixed. Then* M *is unique if and only if* $\operatorname{rank}(A) = r$*.*

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Proof.

For sufficiency, first assume $\operatorname{rank}(A) = r$. Let $M/A = D - CA^{\dagger}B$ be the generalized Schur complement.

Then we have $\operatorname{rank}(M/A) + \operatorname{rank}(A) = \operatorname{rank}(M)$. Therefore $\operatorname{rank}(M/A) = 0$ and $D = CA^{\dagger}B$ is unique. □

Proof.

For necessity, assume $\text{rank}(A) < r$, assume $\text{Range}(B) \subseteq \text{Range}(A)$ and $\text{Range}(C^T) \subseteq \text{Range}(A^T)$. Then by Lemma 1 we have the following equality

$$\text{rank}(D - CA^{\dagger}B) + \text{rank}(A) = \text{rank}(M)$$

Since $\text{rank}(A) < r$ and $\text{rank}(M) = r$, we have $\text{rank}(D - CA^{\dagger}B) = \text{rank}(M) - \text{rank}(A) = \bar{r} > 0$. Let $D = CA^{\dagger}B + E$ where E is an arbitrary matrix of rank $\bar{r}$. Therefore D is not unique.

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1. Now suppose either $\text{Range}(B) \nsubseteq \text{Range}(A)$ or $\text{Range}(C^T) \nsubseteq \text{Range}(A^T)$. Without loss assume $\text{Range}(C^T) \nsubseteq \text{Range}(A^T)$.
2. Then $\text{Null}(A) \nsubseteq \text{Null}(C)$ which means there exists a column vector x such that $Ax = 0, Cx \neq 0$.
3. Now the column vector $\begin{bmatrix} Ax \\ Cx \end{bmatrix} = \begin{bmatrix} 0 \\ Cx \end{bmatrix}$ to any column of $\begin{pmatrix} B \\ D \end{pmatrix}$ without changing the rank of M

□

Corollary 3

Let $Z, \operatorname{rank}(Z) = r$, *be the following matrix with two intersecting bicliques and corresponding submatrices* X *and* Y *which are fixed,*

$$Z = \left[\begin{array}{c|cc} Z_1 & X_1 & X_2 \\ \hline Y_1 & \multicolumn{1}{|c|}{Q} & X_3 \\ \cline{2-3} Y_2 & Y_3 & \multicolumn{1}{|c}{Z_2} \end{array}\right], \quad X = \left[\begin{array}{cc} X_1 & X_2 \\ \cline{1-1} \multicolumn{1}{c|}{Q} & X_3 \end{array}\right], \quad Y = \left[\begin{array}{c|c} Y_1 & Q \\ \cline{2-2} Y_2 & Y_3 \end{array}\right]. \tag{2}$$

submatrix Q *is the part that lies in both* X *and* Y. *If* $\operatorname{rank}(Q) = r$, *then* Z *is unique.*

Proof.

Simple. □

However, the necessity may not be true. Consider the following example

$$Z = \left[\begin{array}{c|cc|c} Z_1 & 6 & 5 & 3 \\ \hline 1 & 2 & 3 & 2 \\ \cline{2-4} 3 & 4 & 2 & Z_2 \end{array} \right], \quad Q = \left[\begin{array}{cc} 2 & 3 \end{array} \right], \tag{3}$$

Assume $\text{rank}(Z) = 2$ and $\text{rank}(Q) = 1 < \text{rank}(Z)$. However, Z_1 and Z_2 are still unique and by basic linear algbera we have $Z_1 = 4$ and $Z_2 = 1$.

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Later we will show that this is the case for positive semidefinite matrices!

Generalization

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Generalization

1. From Theorem 2, if we have two bicliques such that their intersection has the target rank, we can now merge these two bicliques into one bigger biclique and recover the corresponding missing entries of Z.
2. We can then use this bigger biclique to merge with other bicliques.
3. This process can carry on until all the missing entries are recovered given enough bicliques.

Theorem 4

Consider the partitioned matrix

$$M = \begin{bmatrix} E & F \\ A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{m \times n},$$

where $\operatorname{rank}(M) = r$, *and the blocks* A, B, C, F *are fixed. Then* M *is unique if and only if* $\operatorname{rank}(A) = r$ *and* $\operatorname{rank}(B) = r$.

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Proof.

Suppose $\operatorname{rank}(A) = \operatorname{rank}(B) = r$, then it is obvious that D, E are unique by Theorem 2.

For necessity, without loss we assume $\operatorname{rank}(A) = \bar{r} < r$. By a permutation, let

$$M = \begin{bmatrix} A & B \\ E & F \\ C & D \end{bmatrix}.$$

If $\operatorname{rank}(\begin{bmatrix} A \\ E \end{bmatrix}) < r$, then by Therem 2, D is not unique so M is not unique. □

Proof.

If $\text{rank}(\begin{bmatrix} A \\ E \end{bmatrix}) = r$, then we have $\text{Range}(B) \subseteq \text{Range}(A)$. Now we partition A, E, C such that

$$M = \begin{bmatrix} A_1 & A_2 & B \\ E_1 & E_2 & F \\ C_1 & C_2 & D \end{bmatrix}$$

where A_1 has full column rank $\bar{r}$. So we have $\text{Range}(A_1) = \text{Range}(A)$.

Let M_1, M_2, M_3, M_4 be the four Schur complements corresponding to E_2, F, C_2, D such that $M_1 = E_2 - E_1 A_1^{\dagger} A_2$, $M_2 = F - E_1 A_1^{\dagger} B$, $M_3 = C_2 - C_1 A_1^{\dagger} A_2$, $M_4 = D - C_1 A_1^{\dagger} B$.

Proof.

If $\mathrm{rank}(\begin{bmatrix} A \\ E \end{bmatrix}) = r$, then we have $\mathrm{Range}(B) \subseteq \mathrm{Range}(A)$. Now we partition A, E, C such that

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Since $\mathrm{Range}(A_2) \subseteq \mathrm{Range}(A_1)$, $\mathrm{Range}(B) \subseteq \mathrm{Range}(A_1)$ and $\mathrm{Range}(E_1^T) \subseteq \mathrm{Range}(A_1^T)$, $\mathrm{Range}(C_1^T) \subseteq \mathrm{Range}(A_1^T)$, we have

$$\mathrm{rank}(M) = \mathrm{rank}(A_1) + \mathrm{rank}(\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}) = r.$$

□

Also

$$\mathrm{rank}(\begin{bmatrix} A \\ E \end{bmatrix}) = \mathrm{rank}(A_1) + \mathrm{rank}(M_1) = r.$$

Therefore $\mathrm{rank}(M_1) = \mathrm{rank}(\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}) = r - \bar{r}$ and we have

$$M_4 = M_3 M_1^{\dagger} M_2. \tag{4}$$

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Now $M_1 \neq 0$, since $\mathrm{rank}(M_1) = r - \bar{r} > 0$, we can perturb E_2 such that $\bar{E}_2 = E_2 + M_1$ and perturb D such that $\bar{D} = D - \frac{1}{2} M_4$ and the corresponding full perturbed matrix is $\bar{M}$. After similar arguments we can get

$$\begin{aligned} \mathrm{rank}(\bar{M}) &= \mathrm{rank}(A_1) + \mathrm{rank}(\begin{bmatrix} 2M_1 & M_2 \\ M_3 & \frac{1}{2} M_4 \end{bmatrix}) \\ &= \mathrm{rank}(A_1) + \mathrm{rank}(2M_1) + \mathrm{rank}(\frac{1}{2} M_4 - M_3 (2M_1)^{\dagger} M_2) \\ &= \mathrm{rank}(A_1) + \mathrm{rank}(2M_1) \quad \text{(due to (4))} \\ &= \bar{r} + r - \bar{r} = r. \end{aligned}$$

Therefore M is not unique.

The more general case is also true:

Theorem 5

Consider the partitioned matrix

$$M = \begin{bmatrix} F & H & E \\ A & G & B \\ C & K & D \end{bmatrix} \in \mathbb{R}^{m\times n},$$

where $\operatorname{rank}(M) = r$*, and the blocks* A, B, C, E, G *are fixed. Then the matrix* M *is unique if and only if* $\operatorname{rank}(A) = r$ *and* $\operatorname{rank}(B) = r$*.*

Proof.

Without loss we assume $\operatorname{rank}(A) < r$. Now let $B = [G,\ B]$, $E = [H,\ E]$, the result follows directly from Theorem 4. $\square$

Theorem 6

Consider the partitioned matrix

$$M = \begin{bmatrix} F & H & E \\ A & G & B \\ C & K & D \end{bmatrix} \in \mathbb{R}^{m\times n},$$

where $\operatorname{rank}(M) = r$*, and the blocks* F, A, G, K, D *are fixed. Then the matrix* M *is unique if and only if* $\operatorname{rank}(A) = \operatorname{rank}(G) = \operatorname{rank}(K) = r$.

Theorem 6

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Proof.

If $\mathrm{rank}(A) < r$ or $\mathrm{rank}(K) < r$, then according to Theorem 4, M is not unique. Therefore we only need to consider the case when $\mathrm{rank}(G) < r$. If $\mathrm{rank}(\begin{bmatrix} H \\ G \end{bmatrix}) < r$ or $\mathrm{rank}(\begin{bmatrix} G & B \end{bmatrix}) < r$, then again according to Theorem 4, M is not unique. Therefore we consider the case where $\mathrm{rank}(\begin{bmatrix} H \\ G \end{bmatrix}) = r$ and $\mathrm{rank}(\begin{bmatrix} G & B \end{bmatrix}) = r$. □

Proof.

By a permutation, let

$$M = \begin{bmatrix} G & B & A \\ H & E & F \\ K & D & C \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

Let $P = \begin{bmatrix} G & B \\ H & E \end{bmatrix}$, since $\mathrm{rank}(G) < r$, by Theorem 2, there exists a different $\bar{E}$ and $\bar{P}$ such that $\mathrm{rank}(\bar{P}) = r$, we let $\bar{C} = \begin{bmatrix} K & D \end{bmatrix} \bar{P}^{\dagger} \begin{bmatrix} A \\ F \end{bmatrix})$, since $\mathrm{Range}(\begin{bmatrix} A \\ F \end{bmatrix}) \subseteq \mathrm{Range}(\begin{bmatrix} G \\ H \end{bmatrix})$ and $\mathrm{Range}(\begin{bmatrix} K^T \\ D^T \end{bmatrix}) \subseteq \mathrm{Range}(\begin{bmatrix} G^T \\ B^T \end{bmatrix})$, we have $\mathrm{rank}(\bar{M}) = \mathrm{rank}(\bar{P}) + \mathrm{rank}(\bar{C} - \begin{bmatrix} K & D \end{bmatrix} \bar{P}^{\dagger} \begin{bmatrix} A \\ F \end{bmatrix}) = \mathrm{rank}(\bar{P}) = r$.

The corresponding $\bar{M}$ is different from M and the proof is finished. $\square$

Theorem 7

Consider the partitioned matrix

$$M = \begin{bmatrix} F & H & E \\ A & G & B \\ C & K & D \\ J & I & L \end{bmatrix} \in \mathbb{R}^{m\times n},$$

where $\operatorname{rank}(M) = r$*, and the blocks* F, A, G, K, D, L *are fixed. Then the matrix* M *is unique if and only if* $\operatorname{rank}(A) = \operatorname{rank}(G) = \operatorname{rank}(K) = \operatorname{rank}(D) = r$.

Proof.

Direct consequences from Theorem 4 and Theorem 6. □

The above arguments can be extended into a more general case where we have a stair case of known block matrices. We conclude that the whole matrix is unique if and only if every “corner" matrix has rank r.

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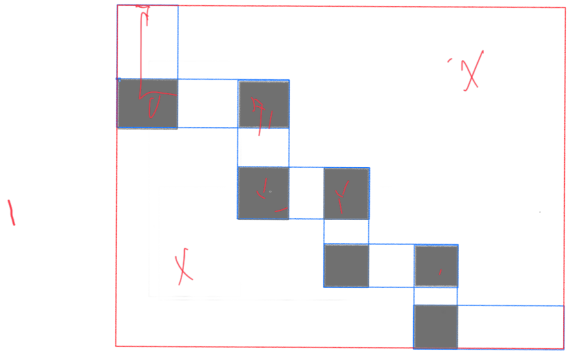
Theorem 8

Given a low rank matrix $Z \in \mathbb{R}^{m\times n}$ and a partial sampling $\mathcal{P}_{\hat{E}}(Z) = z$. If by a permutation there exists a chain of bicliques $\alpha_1, ..., \alpha_l$ with the corresponding edge sets $E_1, \cdots, E_l$. Assume $\cup_{i=1}^{l} E_i = \hat{E}$ and for any i we have $E_i \cap E_j = \emptyset \ \forall j > i+2 \mod l$ and the union of all the vertices of the bicliques satisfy $\cup_{i=1}^{l} \alpha_i = \{1, ..., m\} \times \{1, ..., n\}$. Then the matrix Z can be uniquely recovered if and only if $\operatorname{rank}(X_{\alpha_i \cap \alpha_{i+1}}) = r, i = 1, ..., l-1$

Figures



Matrix is uniquely completable if and only if all gray matrices have the same rank as the big matrix.

Positive semidefinite matrices

We recall the following theorem about symmetric matrix:

Theorem 9

Suppose M is symmetric and partitioned as

$$M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

in which A and C are square. Then $M \succeq 0$ if and only if $A \succeq 0, \mathrm{Range}(B) \subseteq \mathrm{Range}(A)$, and $M/A \succeq 0$.

Theorem 10

Consider the partitioned matrix

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in \mathbb{R}^{m \times n},$$

where $\operatorname{rank}(M) = r$ *and* $M \succeq 0$ *in which* A, C *are square, and the blocks* A, B *are fixed. Then there exists a unique positive semidefinite matrix* M *if and only if* $\operatorname{rank}(A) = r$.

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Proof.

For necessity, assume $\operatorname{rank}(A) < r$, the existence of $M \succeq 0$ ensures $\operatorname{Range}(B) \subseteq \operatorname{Range}(A)$ and $A \succeq 0$, therefore

M/A

$$\operatorname{rank}(M/A) + \operatorname{rank}(A) = \operatorname{rank}(M)$$

Since $\operatorname{rank}(A) < r$ and $\operatorname{rank}(M) = r$, we have $\operatorname{rank}(C - B^T A^\dagger B) = \operatorname{rank}(M) - \operatorname{rank}(A) = \bar{r} > 0$. We can then let $C = B^T A^\dagger B + E$ where E is an arbitrary positive semidefinite matrix of rank $\bar{r}$.

$\square$

Proof.

Hence by Theorem 9

$$\bar{M} = \begin{bmatrix} A & B \\ B^T & B^T A^{\dagger} B + E \end{bmatrix} \succeq 0.$$

Therefore M is not unique. $\square$

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$$\bar{M} = \begin{bmatrix} A & B \\ B^T & B^T A^\dagger B + E \end{bmatrix} \succeq 0.$$

Therefore M is not unique. $\square$

Theorem 11

Consider the partitioned matrix

$$M = \begin{bmatrix} A & B & D \\ B^T & C & E \\ D^T & E^T & F \end{bmatrix} \in \mathbb{R}^{m \times n},$$

M is symmetric and positive semidefinite and the diagonal elements of M are all nonzeros. Suppose A, B, C, E, F are fixed and $\operatorname{rank}(M) = r$*. Then M can be uniquely completed if and only if* $\operatorname{rank}(C) = r$.

Proof.

The if part is obvious, now we prove the only if part.

Let $H = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ and assume $\text{rank}(\begin{bmatrix} C \end{bmatrix}) < r$ and $\text{rank}(H) = r$.

Let $\tilde{D} = D + X$, then by Schur complement, $X = 0$ is a solution of the equation

$$F - [\tilde{D}^T, E^T] H^\dagger \begin{bmatrix} \tilde{D} \\ E \end{bmatrix} = 0. \tag{5}$$

$\begin{bmatrix} X \\ 0 \end{bmatrix} \in \text{Rang}\begin{pmatrix} A \\ B^T \end{pmatrix}$

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Therefore this equation is homogeneous and we can assume (5) has the following form:

$$[X^T, 0](\frac{1}{2}H^\dagger \begin{bmatrix} X \\ 0 \end{bmatrix} + H^\dagger \begin{bmatrix} D \\ E \end{bmatrix}) + ([X^T, 0]\frac{1}{2}H^\dagger + \begin{bmatrix} D^T & E^T \end{bmatrix} H^\dagger) \begin{bmatrix} X \\ 0 \end{bmatrix} \tag{6}$$

Proof.

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Let $H = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ and assume $\text{rank}(\begin{bmatrix} C \end{bmatrix}) < r$ and $\text{rank}(H) = r$.

Let $\tilde{D} = D + X$, then by Schur complement, $X = 0$ is a solution of the equation

$$F - [\tilde{D}^T, E^T] H^\dagger \begin{bmatrix} \tilde{D} \\ E \end{bmatrix} = 0. \tag{5}$$

Therefore this equation is homogeneous and we can assume (5) has the following form:

$[X^\dagger, 0]^T \begin{bmatrix} 0 \\ Z \end{bmatrix} = 0$

$$[X^T, 0]\left(\frac{1}{2} H^\dagger \begin{bmatrix} X \\ 0 \end{bmatrix} + H^\dagger \begin{bmatrix} D \\ E \end{bmatrix}\right) + \left([X^T, 0]\frac{1}{2} H^\dagger + \begin{bmatrix} D^T & E^T \end{bmatrix} H^\dagger\right) \begin{bmatrix} X \\ 0 \end{bmatrix} \tag{6}$$

Let $H^\dagger = \begin{bmatrix} A^\dagger & B^\dagger \\ (B^\dagger)^T & C^\dagger \end{bmatrix}$ and consider $H^\dagger \begin{bmatrix} X \\ 0 \end{bmatrix} + 2H^\dagger \begin{bmatrix} D \\ E \end{bmatrix}$. Clearly we only need to require $A^\dagger X - 2(A^\dagger D + BE) = 0$ such that equation (6) holds true.

$\square$

Proof.

1. $A^\dagger D + \bar{B}E = 0$ implies $D = E = 0$ since $\begin{bmatrix} D & E \end{bmatrix} \in \text{Range}(H) = \text{Range}(H^\dagger)$.
2. It implies $F = 0$ since $\text{rank}(H) = \text{rank}(M) = r$ which contradicts our assumption that the diagonal elements are all nonzeros.
3. Let $2(A^\dagger D + B^\dagger E) = G$, then $G \neq 0$ and $X = AG \neq 0$ is a solution.

Proof.

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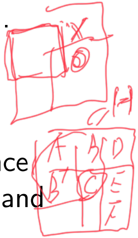
3. Let $2(A^\dagger D + B^\dagger E) = G$, then $G \neq 0$ and $X = AG \neq 0$ is a solution.

Now we need to show that $\begin{bmatrix} X \\ 0 \end{bmatrix} \in \text{Range}(H) = \text{Range}(H^\dagger)$.



1. This is true if $\text{Range}(\begin{bmatrix} A \\ B^T \end{bmatrix}) \cap \text{Range}(\begin{bmatrix} B \\ C \end{bmatrix}) = \{0\}$.
2. This assumption may not hold in general, however, since $\text{rank}(C) < \text{rank}(H)$, we can perform symmetrical row and column elementary operations such that it holds.
3. After elementary row and column operations and a nonzero X is found, we can do reverse operations such that the fixed elements of M stay the same.

Theorem 12

Suppose the observed entries of a positive semidefinite matrix of rank r are blocked diagonal with overlapping matrices. Then matrix M can be uniquely recovered if and only if the overlapping matrices also have rank r.

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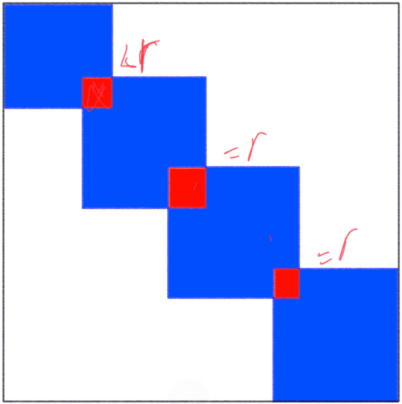

Future work

1. Algorithms to permute matrices to blocked diagonal with overlapping submatrices.
2. Connection to rigidity theory: The Lindenstrauss mapping, $\mathcal{K} : \mathcal{S}_n \to \mathcal{S}_n$,

$$\mathcal{K}(X)_{ij} := X_{ii} + X_{jj} - 2X_{ij}. = D$$

Bibliography I

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Thank You for Your Attention!