Cofactor Matroids and Abstract Rigidity

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A matroid $\mathcal{M}$ is a pair $(E, I)$ where $E$ is a finite set and $I$ is a family of subsets of $E$ satisfying:

- $\emptyset \in I$;
- if $A \subseteq B \subseteq E$ and $B \in I$ then $A \in I$;
- if $A, B \in I$ and $|A| < |B|$ then there exists $x \in B \setminus A$ such that $A + x \in I$.
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$A \subseteq E$ is **independent** if $A \in \mathcal{I}$ and $A$ is **dependent** if $A \notin \mathcal{I}$. The minimal dependent sets of $\mathcal{M}$ are the **circuits** of $\mathcal{M}$. The **rank** of $A$, $r(A)$, is the cardinality of a maximal independent subset of $A$. The **rank** of $\mathcal{M}$ is the cardinality of a maximal independent subset of $E$. 

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**Cofactor Matroids**

A **cofactor matroid** is a matroid with the property that for every edge $e$, the matroid obtained by deleting $e$ and then adding a new element $v$ and connecting it to all previously independent elements is isomorphic to the original matroid. This property is useful in the study of abstract rigidity.

**Abstract Rigidity**

In the context of matroids, abstract rigidity refers to the property of a matroid that ensures the structure is rigid, meaning it cannot be deformed into a different configuration without breaking any of its fundamental properties. This concept is crucial in the study of structural stability in various applications, including engineering and computer science.
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The **weak order** on a set $S$ of matroids with the same groundset is defined as follows. Given two matroids $\mathcal{M}_1 = (E, I_1)$ and $\mathcal{M}_2 = (E, I_2)$ in $S$, we say $\mathcal{M}_1 \preceq \mathcal{M}_2$ if $I_1 \subseteq I_2$. 
A \textbf{d-dimensional framework} \((G, p)\) is a graph \(G = (V, E)\) together with a map \(p : V \to \mathbb{R}^d\).
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The \textbf{rigidity matrix} of \((G, p)\) is the matrix \(R(G, p)\) of size \(|E| \times d|V|\) in which the row associated with the edge \(v_i v_j\) is

\[
\begin{bmatrix}
v_i & & v_j \\
0 & \ldots & 0 & p(v_i) - p(v_j) & 0 & \ldots & 0 & p(v_j) - p(v_i) & 0 & \ldots & 0
\end{bmatrix}.
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The generic $d$-dimensional rigidity matroid

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The **generic $d$-dimensional rigidity matroid** $\mathcal{R}_{n,d}$ is the row matroid of the rigidity matrix $R(K_n, p)$ for any generic $p : V(K_n) \to \mathbb{R}^d$. 

The generic $d$-dimensional rigidity matroid

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$$v_i v_j \begin{bmatrix} 0 \cdots 0 & p(v_i) - p(v_j) & 0 \cdots 0 & p(v_j) - p(v_i) & 0 \cdots 0 \end{bmatrix}. $$

The generic $d$-dimensional rigidity matroid $R_{n,d}$ is the row matroid of the rigidity matrix $R(K_n, p)$ for any generic $p : V(K_n) \rightarrow \mathbb{R}^d$. $R_{n,d}$ is a matroid with groundset $E(K_n)$ with rank $dn - \binom{d+1}{2}$. It is the algebraic matroid of the $d$-dimensional Cayley-Hamilton variety defined by the polynomial equations $\|p(v_i) - p(v_j)\|^2 = d_{ij}$. Its rank function can be determined (by good characterisations and polynomial algorithms) when $d = 1, 2$. Obtaining such characterisations for $d \geq 3$ is a long standing open problem.
Abstract $d$-rigidity matroids

Jack Graver (1991) chose two closure properties of $\mathcal{R}_{d,n}$ and used them to define the family of abstract $d$-rigidity matroids on $E(K_n)$. Viet Hang Nguyen (2010) gave the following equivalent definition: $\mathcal{M}$ is an abstract $d$-rigidity matroid iff $\text{rank } \mathcal{M} = dn - \binom{d+1}{2}$, and every $K_{d+2} \subseteq K_n$ is a circuit in $\mathcal{M}$. 

Conjecture [Graver, 1991] For all $d$, $n \geq 1$, $\mathcal{R}_{d,n}$ is the unique maximal element in the family of all abstract $d$-rigidity matroids on $E(K_n)$.

Graver verified his conjecture for $d = 1, 2$. Walter Whiteley (1996) gave counterexamples to Graver’s conjecture for all $d \geq 4$ and $n \geq d+2$ using ‘cofactor matroids’.
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Given a polygonal subdivision $\Delta$ of a polygonal domain $D$ in the plane, a bivariate function $f : D \rightarrow \mathbb{R}$ is an $(s, k)$-spline over $\Delta$ if it is defined as a polynomial of degree $s$ on each face of $\Delta$ and is continuously differentiable $k$ times on $D$. 

The set $S_{k,s}(\Delta)$ of $(s, k)$-splines over $\Delta$ forms a vector space. Obtaining tight upper/lower bounds on $\dim S_{k,s}(\Delta)$ (over a given class of subdivisions $\Delta$) is an important problem in approximation theory. Whiteley (1990) observed that $\dim S_{k,s}(\Delta)$ can be calculated from the rank of a matrix $C_{k,s}(G, p)$ which is determined by the 1-skeleton $(G, p)$ of the subdivision $\Delta$ (viewed as a 2-dim framework), and that rigidity theory can be used to investigate the rank of this matrix. His definition of $C_{k,s}(G, p)$ makes sense for all 2-dim frameworks (not just frameworks whose underlying graph is planar).
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Bivariate Splines and Cofactor Matrices

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- His definition of $C^k_s(G, p)$ makes sense for all 2-dim frameworks (not just frameworks whose underlying graph is planar).
Let \((G, p)\) be a 2-dimensional framework and put \(p(v_i) = (x_i, y_i)\) for \(v_i \in V(G)\). For \(v_i v_j \in E(G)\) and \(d \geq 1\) let
\[
D_d(v_i, v_j) = ((x_i - x_j)^{d-1}, (x_i - x_j)^{d-2}(y_i - y_j), \ldots, (y_i - y_j)^{d-1}).
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The \(C^{d-2}_{d-1}\)-cofactor matrix of \((G, p)\) is the matrix \(C^{d-2}_{d-1}(G, p)\) of size \(|E| \times d|V|\) in which the row associated with the edge \(v_i v_j\) is
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v_i v_j \begin{bmatrix}
0 & \ldots & 0 & D_d(v_i, v_j) & 0 & \ldots & 0 & -D_d(v_i, v_j) & 0 & \ldots & 0
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Let \((G, p)\) be a 2-dimensional framework and put \(p(v_i) = (x_i, y_i)\) for \(v_i \in V(G)\). For \(v_iv_j \in E(G)\) and \(d \geq 1\) let
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\]
The \(C_{d-1}^{d-2}\)-cofactor matrix of \((G, p)\) is the matrix \(C_{d-1}^{d-2}(G, p)\) of size \(|E| \times d|V|\) in which the row associated with the edge \(v_iv_j\) is
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\begin{bmatrix}
v_i
D_d(v_i, v_j) & 0 & \ldots & 0 & -D_d(v_i, v_j) & 0 & \ldots & 0
\end{bmatrix}.
\]
The generic \(C_{d-1}^{d-2}\)-cofactor matroid, \(C_{d-1}^{d-2}(K_n, p)\) is the row matroid of the cofactor matrix \(C_{d-1}^{d-2}(K_n, p)\) for any generic \(p\).
Theorem [Whiteley]

- $C^{d-2}_{d-1,n}$ is an abstract $d$-rigidity matroid for all $d, n \geq 1$.
- $C^{d-2}_{d-1,n} = \mathcal{R}_{d,n}$ for $d = 1, 2$.
- $C^{d-2}_{d-1,n} \not\subseteq \mathcal{R}_{d,n}$ when $d \geq 4$ and $n \geq 2(d + 2)$ since $K_{d+2,d+2}$ is independent in $C^{d-2}_{d-1,n}$ and dependent in $\mathcal{R}_{d,n}$.
**Theorem [Whiteley]**

- $C_{d-1,n}^{d-2}$ is an abstract $d$-rigidity matroid for all $d, n \geq 1$.
- $C_{d-1,n}^{d-2} = R_{d,n}$ for $d = 1, 2$.
- $C_{d-1,n}^{d-2} \not\preceq R_{d,n}$ when $d \geq 4$ and $n \geq 2(d + 2)$ since $K_{d+2,d+2}$ is independent in $C_{d-1,n}^{d-2}$ and dependent in $R_{d,n}$.

**Conjecture [Whiteley, 1996]**

For all $d, n \geq 1$, $C_{d-1,n}^{d-2}$ is the unique maximal abstract $d$-rigidity matroid on $E(K_n)$. 
Theorem [Whiteley]

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For all $n \geq 1$, $C_{2,n}^1 = \mathcal{R}_{3,n}$. 
The maximal abstract 3-rigidity matroid

**Theorem [Clinch, BJ, Tanigawa 2019+]**

$C_{2,n}^1$ is the unique maximal abstract 3-rigidity matroid on $E(K_n)$.

**Sketch Proof**

Suppose $M$ is an abstract 3-rigidity matroid on $E(K_n)$ and $F \subseteq E(K_n)$ is independent in $M$. We show that $F$ is independent in $C_{2,n}^1$ by induction on $|F|$. Since $M$ is an abstract 3-rigidity matroid, $|F| = r(F) \leq 3|V(F)| - 6$ and hence $F$ has a vertex $v$ with $d_F(v) \leq 5$. In this case, we extend $F$ to an independent set $F'$ in $M$ by adding $v$ and $d_F'(v) \leq 4$. Then $F'$ is an independent set in $C_{2,n}^1$. Therefore, $F$ is independent in $C_{2,n}^1$. Q.E.D.
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The maximal abstract 3-rigidity matroid

**Theorem [Clinch, BJ, Tanigawa 2019+]**

\( C^1_{2,n} \) is the unique maximal abstract 3-rigidity matroid on \( E(K_n) \).

**Sketch Proof** Suppose \( M \) is an abstract 3-rigidity matroid on \( E(K_n) \) and \( F \subseteq E(K_n) \) is independent in \( M \). We show that \( F \) is independent in \( C^1_{2,n} \) by induction on \( |F| \). Since \( M \) is an abstract 3-rigidity matroid, \( |F| = r(F) \leq 3|V(F)| - 6 \) and hence \( F \) has a vertex \( v \) with \( d_F(v) \leq 5 \).

**Case 1: \( d_F(v) \leq 3 \)**
Theorem [Clinch, BJ, Tanigawa 2019+]

$C^2_{3,n}$ is the unique maximal abstract $d$-rigidity matroid on $E(K_n)$.

**Sketch Proof** Suppose $\mathcal{M}$ is an abstract rigidity matroid on $E(K_n)$ and $F \subseteq E(K_n)$ is independent in $\mathcal{M}$. We show that $F$ is independent in $C^1_{2,n}$ by induction on $|F|$. Since $\mathcal{M}$ is an abstract 3-rigidity matroid, $|F| = r(F) \leq 3|V(F)| - 6$ and hence $F$ has a vertex $v$ with $d_F(v) \leq 5$.

**Case 2:** $d_F(v) = 4$
Case 3: \( d_F(v) = 5 \)

\[ F - v + e + f \]

Independent in \( M \)

Double V-replacement (CJT)

Independent in \( M \)

Independent in \( C_{2,n}^1 \)

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Matroid theory

Induction

Induction
The rank function of $C_{2,n}^1$

A $K_5$-sequence in $K_n$ is a sequence of subgraphs $(K_5^1, K_5^2, \ldots, K_5^t)$ each of which is isomorphic to $K_5$. It is **proper** if $K_5^i \not\subseteq \bigcup_{j=1}^{i-1} K_5^j$ for all $2 \leq i \leq t$. 

Theorem [Clinch, BJ, Tanigawa 2019+]

The rank of any $F \subseteq E(K_n)$ in $C_{2,n}^1$ is given by:

$$r(F) = \min \{|F_0| + \left|\bigcup_{i=1}^t E(K_5^i)\right| - t$$

where the minimum is taken over all $F_0 \subseteq F$ and all proper $K_5$-sequences $(K_5^1, K_5^2, \ldots, K_5^t)$ in $K_n$ which cover $F \setminus F_0$. 

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Cofactor Matroids and Abstract Rigidity
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where the minimum is taken over all $F_0 \subseteq F$ and all proper $K_5$-sequences $(K_5^1, K_5^2, \ldots, K_5^t)$ in $K_n$ which cover $F \setminus F_0$. 
Let $F_0 = \{e_1, e_2, e_3\}$ and $(K_5^1, K_5^2, \ldots, K_5^7)$ be the ‘obvious’ proper $K_5$-sequence which covers $F \setminus F_0$. We have $|F| = 60$ and

$$r(F) \leq |F_0| + \left| \bigcup_{i=1}^{7} E(K_5^i) \right| - 7 = 59$$

so $F$ is not independent in $C^1_{2,n}$. Since $3|V(F)| - 6 = 60$, $F$ is not rigid in any abstract 3-rigidity matroid.
Theorem [Clinch, BJ, Tanigawa 2019+]  
Let $\mathcal{M}$ be a matroid, $\mathcal{M}_0$ be the truncation of $\mathcal{M}$ to rank $k$ and $S$ be the set of all matroids which can be truncated to $\mathcal{M}_0$. Suppose that $\mathcal{M}$ is the unique maximal matroid in $S$ and $F$ is a cyclic flat in $\mathcal{M}$. Then every element of $F$ belongs to a circuit of $\mathcal{M}_0$ in $F$.

Take $\mathcal{M} = C_{2,n}^1$ and $k = 10$.

Corollary

Suppose $F \subseteq E(K_n)$ is a cyclic flat in $C_{2,n}^1$. Then every element of $F$ belongs to a copy of $K_5$ in $F$. 

Theorem [Clinch, BJ, Tanigawa 2019+] Every 12-connected graph is rigid in the maximal abstract 3-rigidity matroid.
Theorem [Clinch, BJ, Tanigawa 2019+] 
Every 12-connected graph is rigid in the maximal abstract 3-rigidity matroid.

Lovász and Yemini (1982) conjectured that the analogous result holds for the generic 3-dimensional rigidity matroid. Examples constructed by Lovász and Yemini show that the connectivity hypothesis in the above theorem is best possible.
Problem 1 Determine whether the X-replacement operation preserves independence in the generic 3-dimensional rigidity matroid.
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**Preprints**


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