Cofactor Matroids and Abstract Rigidity

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Algebraic Matroids and Rigidity Theory Seminar Series
25 June, 2020

Matroids

A **matroid** $\mathcal{M}$ is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a family of subsets of E satisfying:

- $\emptyset \in \mathcal{I}$;
- if $A \subseteq B \subseteq E$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$;
- if $A, B \in \mathcal{I}$ and $|A| < |B|$ then there exists $x \in B \setminus A$ such that $A + x \in \mathcal{I}$.

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$A \subseteq E$ is **independent** if $A \in \mathcal{I}$ and A is **dependent** if $A \notin \mathcal{I}$. The minimal dependent sets of $\mathcal{M}$ are the **circuits** of $\mathcal{M}$. The **rank** of A, $r(A)$, is the cardinality of a maximal independent subset of A. The **rank** of $\mathcal{M}$ is the cardinality of a maximal independent subset of E.

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The **weak order** on a set S of matroids with the same groundset is defined as follows. Given two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ in S, we say $\mathcal{M}_1 \preceq M_2$ if $\mathcal{I}_1 \subseteq \mathcal{I}_2$.

The generic d-dimensional rigidity matroid

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The **rigidity matrix** of (G, p) is the matrix $R(G, p)$ of size $|E| \times d|V|$ in which the row associated with the edge $v_i v_j$ is

$$\begin{array}{cccccc} & & v_i & & v_j & \\ v_i v_j & [\; 0 \dots 0 & p(v_i) - p(v_j) & 0 \dots 0 & p(v_j) - p(v_i) & 0 \dots 0 \;]. \end{array}$$

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$\mathcal{R}_{n,d}$ is a matroid with groundset $E(K_n)$ with rank $dn - \binom{d+1}{2}$. It is the algebraic matroid of the d-dimensional **Cayley-Hamilton variety** defined by the polynomial equations $\|p(v_i) - p(v_j)\|^2 = d_{ij}$. Its rank function can be determined (by good characterisations and polynomial algorithms) when $d = 1, 2$. Obtaining such characterisations for $d \geq 3$ is a long standing open problem.

Abstract d-rigidity matroids

Jack Graver (1991) chose two closure properties of $\mathcal{R}_{d,n}$ and used them to define the **family of abstract d-rigidity matroids** on $E(K_n)$. Viet Hang Nguyen (2010) gave the following equivalent definition: $\mathcal{M}$ is an **abstract d-rigidity matroid** iff rank $\mathcal{M} = dn - \binom{d+1}{2}$, and every $K_{d+2} \subseteq K_n$ is a circuit in $\mathcal{M}$.

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Conjecture [Graver, 1991]

For all $d, n \geq 1$, $\mathcal{R}_{d,n}$ is the unique maximal element in the family of all abstract d-rigidity matroids on $E(K_n)$.

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Walter Whiteley (1996) gave counterexamples to Graver's conjecture for all $d \geq 4$ and $n \geq d + 2$ using 'cofactor matroids'.

Bivariate Splines and Cofactor Matrices

Given a polygonal subdivision Δ of a polygonal domain D in the plane, a bivariate function $f : D \to \mathbb{R}$ is an (s, k)-**spline over** Δ if it is defined as a polynomial of degree s on each face of Δ and is continuously differentiable k times on D.

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- The set $S_s^k(\Delta)$ of (s,k)-splines over Δ forms a vector space.
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- Whiteley (1990) observed that $\dim S_s^k(\Delta)$ can be calculated from the rank of a matrix $C_s^k(G, p)$ which is determined by the the 1-skeleton (G, p) of the subdivision Δ (viewed as a 2-dim framework), and that rigidity theory can be used to investigate the rank of this matrix.
- His definition of $C_s^k(G, p)$ makes sense for *all* 2-dim frameworks (not just frameworks whose underlying graph is planar).

Cofactor matroids

Let (G, p) be a 2-dimensional framework and put $p(v_i) = (x_i, y_i)$ for $v_i \in V(G)$. For $v_i v_j \in E(G)$ and $d \geq 1$ let
$D_d(v_i, v_j) = ((x_i - x_j)^{d-1}, (x_i - x_j)^{d-2}(y_i - y_j), \ldots, (y_i - y_j)^{d-1})$.

Cofactor matroids

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The C_{d-1}^{d-2}-**cofactor matrix** of (G, p) is the matrix $C_{d-1}^{d-2}(G, p)$ of size $|E| \times d|V|$ in which the row associated with the edge v_iv_j is

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The generic C_{d-1}^{d-2}-**cofactor matroid**, $\mathcal{C}_{d-1,n}^{d-2}$ is the row matroid of the cofactor matrix $C_{d-1}^{d-2}(K_n,p)$ for any generic p.

Cofactor matroids - Whiteley's Results and Conjectures

Theorem [Whiteley]

- $\mathcal{C}_{d-1,n}^{d-2}$ is an abstract d-rigidity matroid for all $d, n \geq 1$.
- $\mathcal{C}_{d-1,n}^{d-2} = \mathcal{R}_{d,n}$ for $d = 1, 2$.
- $\mathcal{C}_{d-1,n}^{d-2} \not\preceq \mathcal{R}_{d,n}$ when $d \geq 4$ and $n \geq 2(d+2)$ since $K_{d+2,d+2}$ is independent in $\mathcal{C}_{d-1,n}^{d-2}$ and dependent in $\mathcal{R}_{d,n}$.

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Conjecture [Whiteley, 1996]

For all $d, n \geq 1$, $\mathcal{C}_{d-1,n}^{d-2}$ is the unique maximal abstract d-rigidity matroid on $E(K_n)$.

Cofactor matroids - Whiteley's Results and Conjectures

Theorem [Whiteley]

- $\mathcal{C}^{d-2}_{d-1,n}$ is an abstract d-rigidity matroid for all $d, n \geq 1$.
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For all $n \geq 1$, $\mathcal{C}^{1}_{2,n} = \mathcal{R}_{3,n}$.

The maximal abstract 3-rigidity matroid

Theorem [Clinch, BJ, Tanigawa 2019+]

$\mathcal{C}_{2,n}^1$ is the unique maximal abstract 3-rigidity matroid on $E(K_n)$.

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Theorem [Clinch, BJ, Tanigawa 2019+]

$\mathcal{C}^1_{2,n}$ is the unique maximal abstract 3-rigidity matroid on $E(K_n)$.

Sketch Proof Suppose $\mathcal{M}$ is an abstract 3-rigidity matroid on $E(K_n)$ and $F \subseteq E(K_n)$ is independent in M. We show that F is independent in $\mathcal{C}^1_{2,n}$ by induction on $|F|$. Since $\mathcal{M}$ is an abstract 3-rigidity matroid, $|F| = r(F) \leq 3|V(F)| - 6$ and hence F has a vertex v with $d_F(v) \leq 5$.

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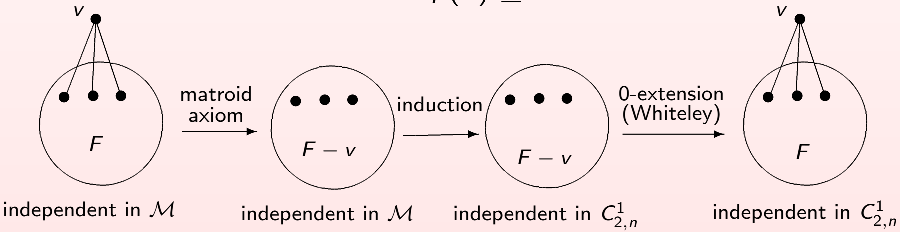
Case 1: $d_F(v) \leq 3$

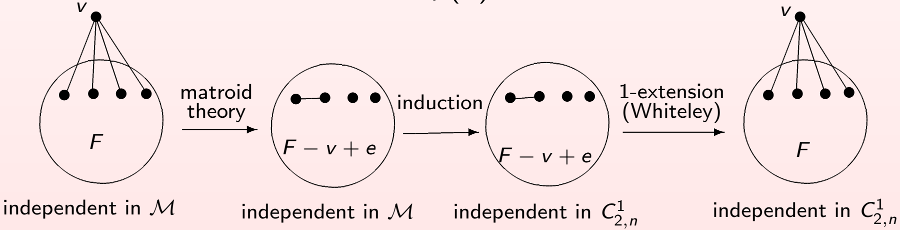


Theorem [Clinch, BJ, Tanigawa 2019+]

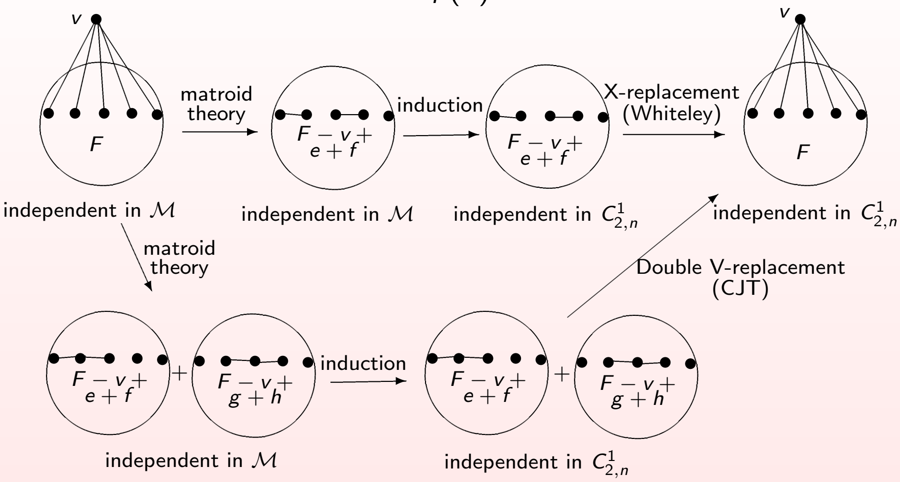
$\mathcal{C}^2_{3,n}$ is the unique maximal abstract d-rigidity matroid on $E(K_n)$.

Sketch Proof Suppose $\mathcal{M}$ is an abstract rigidity matroid on $E(K_n)$ and $F \subseteq E(K_n)$ is independent in M. We show that F is independent in $\mathcal{C}^1_{2,n}$ by induction on $|F|$. Since $\mathcal{M}$ is an abstract 3-rigidity matroid, $|F| = r(F) \leq 3|V(F)| - 6$ and hence F has a vertex v with $d_F(v) \leq 5$.

Case 2: $d_F(v) = 4$



Case 3: $d_F(v) = 5$


v
F
matroid theory
$F - v + e + f$
induction
$F - v + e + f$
X-replacement (Whiteley)
v
F
independent in $\mathcal{M}$
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$+$
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A K_5-**sequence in** K_n is a sequence of subgraphs $(K_5^1, K_5^2, \ldots, K_5^t)$ each of which is isomorphic to K_5.
It is **proper** if $K_5^i \not\subseteq \bigcup_{j=1}^{i-1} K_5^j$ for all $2 \leq i \leq t$.

The rank function of $\mathcal{C}^1_{2,n}$

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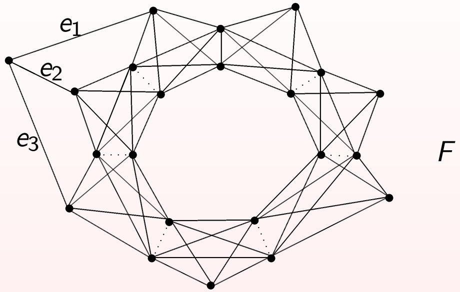
Theorem [Clinch, BJ, Tanigawa 2019+]

The rank of any $F \subseteq E(K_n)$ in $\mathcal{C}^1_{2,n}$ is given by

$$r(F) = \min \left\{ |F_0| + \left| \bigcup_{i=1}^{t} E(K_5^i) \right| - t \right\}$$

where the minimum is taken over all $F_0 \subseteq F$ and all proper K_5-sequences $(K_5^1, K_5^2, \ldots, K_5^t)$ in K_n which cover $F \setminus F_0$.

Example



Let $F_0 = \{e_1, e_2, e_3\}$ and $(K_5^1, K_5^2, \ldots, K_5^7)$ be the 'obvious' proper K_5-sequence which covers $F \setminus F_0$. We have $|F| = 60$ and

$$r(F) \leq |F_0| + \left| \bigcup_{i=1}^{7} E(K_5^i) \right| - 7 = 59$$

so F is not independent in $\mathcal{C}_{2,n}^1$. Since $3|V(F)| - 6 = 60$, F is not rigid in any abstract 3-rigidity matroid.

Key Matroid Lemma

Theorem [Clinch, BJ, Tanigawa 2019+]

Let $\mathcal{M}$ be a matroid, $\mathcal{M}_0$ be the truncation of M to rank k and S be the set of all matroids which can be truncated to $\mathcal{M}_0$. Suppose that $\mathcal{M}$ is the unique maximal matroid in S and F is a cyclic flat in $\mathcal{M}$. Then every element of F belongs to a circuit of $\mathcal{M}_0$ in F.

Take $\mathcal{M} = \mathcal{C}^1_{2,n}$ and $k = 10$.

Corollary

Supose $F \subseteq E(K_n)$ is a cyclic flat in $\mathcal{C}^1_{2,n}$. Then every element of F belongs to a copy of K_5 in F.

Application

Theorem [Clinch, BJ, Tanigawa 2019+]

Every 12-connected graph is rigid in the maximal abstract 3-rigidity matroid.

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Theorem [Clinch, BJ, Tanigawa 2019+]

Every 12-connected graph is rigid in the maximal abstract 3-rigidity matroid.

Lovász and Yemini (1982) conjectured that the analogous result holds for the generic 3-dimensional rigidity matroid. Examples constructed by Lovász and Yemini show that the connectivity hypothesis in the above theorem is best possible.

Open Problems and Preprints

Problem 1 Determine whether the X-replacement operation preserves independence in the generic 3-dimensional rigidity matroid.

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Problem 2 Find a polynomial algorithm for determining the rank function of $\mathcal{C}_{2,n}^1$.

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Preprints

K. Clinch, B. Jackson and S. Tanigawa, Abstract 3-rigidity and bivariate C_2^1-splines I: Whiteley's maximality conjecture, preprint available at https://arxiv.org/abs/1911.00205.

K. Clinch, B. Jackson and S. Tanigawa, Abstract 3-rigidity and bivariate C_2^1-splines II: Combinatorial Characterization, preprint available at https://arxiv.org/abs/1911.00207.