Which graphs are rigid in ℓ_q spaces?

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Normed spaces

Definition

We define a (finite dimensional real) normed space to be a pair $X = (\mathbb{R}^d, \|\cdot\|)$ where $\|\cdot\| : \mathbb{R}^d \to \mathbb{R}$ is a *norm* i.e. for all $x, y \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$:

- $\|x\| \geq 0$ with equality if and only if $x = 0$.
- $\|\lambda x\| = |\lambda| \|x\|$.
- $\|x + y\| \leq \|x\| + \|y\|$.

- $\ell_q^d := (\mathbb{R}^d, \|\cdot\|_q)$, $q \in [1, \infty)$,

$$\|(x_1, \ldots, x_d)\|_q := \left(\sum_{i=1}^{d} |x_i|^q \right)^{1/q}.$$

- $\ell_\infty^d := (\mathbb{R}^d, \|\cdot\|_\infty)$, $\|(x_1, \ldots, x_d)\|_\infty := \max\{|x_1|, \ldots, |x_d|\}$.

Differentiating the norm

Definition

A point $x \in X \setminus \{0\}$ is *smooth* if the norm is differentiable there. We denote the derivative of $\frac{1}{2}\|\cdot\|^2$ at x by $\varphi_x \in X^*$; note that $\varphi_0 = 0$.

- φ_x is also the unique *support functional* of x, i.e. a linear functional where $\varphi_x(x) = \|x\|^2$ and $\|\varphi_x\|^* = \|x\|$.
- Almost all points are smooth.
- The map $x \mapsto \varphi_x$ is continuous on the smooth points plus 0; also linear if and only if X is Euclidean.

Example (ℓ_q^d for $q \in (1, \infty)$)

$\varphi_x(y) := cx^{(q-1)}.y$, where $x^{(q-1)} := (\mathrm{sgn}(x_1)|x_1|^{q-1}, \dots, \mathrm{sgn}(x_d)|x_d|^{q-1})$ and $c = \|x\|_q^{2-q}$.

Example (ℓ_∞^d)

$\varphi_x(y) := x_i y_i$, where $|x_i| > |x_j|$ for all $j \neq i$.

Rigidity matrix and independence

Let (G, p) be a *(well-positioned) framework* in X, i.e. $G = (V, E)$ (finite simple) graph, $p : V \to X$, $p_v - p_w$ smooth for every $vw \in E$. The *rigidity matrix of* (G, p) with respect to a basis $b_1, \ldots, b_d \in X$ is the $|E| \times d|V|$ real valued matrix $R(G, p)$ with entries

$$r_{e,(v,k)} = \begin{cases} \varphi_{p_v - p_w}(b_k) & \text{if } e = vw, \\ 0 & \text{otherwise.} \end{cases}$$

For ℓ_q^d we simplify; the *altered rigidity matrix of* (G, p) is the $|E| \times d|V|$ real valued matrix $\tilde{R}(G, p)$ with entries

$$r_{e,(v,k)} = \begin{cases} [(p_v - p_w)^{(q-1)}]_k & \text{if } e = vw, \\ 0 & \text{otherwise.} \end{cases}$$

We say (G, p) is *independent* if $\operatorname{rank} R(G, p) = |E|$.

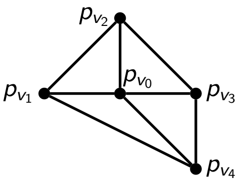
Rigidity matrix example for ℓ_q^2

Define p to be the placement of the wheel graph W_5 with center v_0 in ℓ_q^2 where,

$$p_{v_0} = (0,0), \quad p_{v_1} = (-1,0), \quad p_{v_2} = (0,1), \quad p_{v_3} = (1,0), \quad p_{v_4} = (1,-1).$$

The altered rigidity matrix $\tilde{R}(W_5, p)$:

	$(v_0,1)$	$(v_0,2)$	$(v_1,1)$	$(v_1,2)$	$(v_2,1)$	$(v_2,2)$	$(v_3,1)$	$(v_3,2)$	$(v_4,1)$	$(v_4,2)$
v_0v_1	1	0	-1	0	0	0	0	0	0	0
v_0v_2	0	-1	0	0	0	1	0	0	0	0
v_0v_3	-1	0	0	0	0	0	1	0	0	0
v_0v_4	-1	1	0	0	0	0	0	0	1	-1
v_1v_2	0	0	-1	-1	1	1	0	0	0	0
v_2v_3	0	0	0	0	-1	1	1	-1	0	0
v_3v_4	0	0	0	0	0	0	0	1	0	-1
v_1v_4	0	0	-2^{q-1}	1	0	0	0	0	2^{q-1}	-1



For $q \neq 2$, rank $\tilde{R}(W_5, p) = |E|$, hence (W_5, p) is independent in ℓ_q^2.

Rigidity

Let (G, p) be *regular* in X[1], i.e. $\operatorname{rank} R(G, p) \geq \operatorname{rank} R(G, p')$ for all other well-positioned frameworks. (G, p) is *rigid* if one of the following equivalent conditions hold (and is *flexible* otherwise):

(i) If $\gamma : [0, 1] \to X^V$ is a continuous path with $\gamma(0) = p$, $\gamma(1) = p'$ and

$$\|\gamma(t)_v - \gamma(t)_w\| = \|p_v - p_w\| \qquad \text{for all } t \in [0, 1],\ vw \in E$$

then (G, p) and (G, p') are isometric.

(ii) There exists an open neighbourhood $U \subset X^V$ of p such that if $p' \in U$ and $\|p'_v - p'_w\| = \|p_v - p_w\|$ for all $vw \in E$, then (G, p) and (G, p') are isometric.

(iii) $\operatorname{rank} R(G, p) = d|V| - k(X)$, where $k(X)$ denotes the dimension of the isometry group of X.

G is *rigid* (resp. *independent*, *flexible*) in X if there exists a rigid (resp. independent, flexible) regular framework (G, p) in X. $(G, p)/G$ is *minimally rigid* if it is both independent and rigid.

[1] We also make the assumption that the set of smooth points of X is open. This is a natural assumption to make as all of the classical normed spaces have this property, and those that don't form a meager subset of the set of all normed spaces.

So what is $k(X)$?

For $q \in [1,2) \cup (2,\infty]$, $k(\ell_q^d) = d$, as only isometries are signed permutation matrices + translations.

Theorem (Montgomery and Samelson (1943))

If X is Euclidean then $k(X) = \binom{d+1}{2}$, while if X is non-Euclidean then $d \leq k(X) \leq \binom{d}{2} + 1$.

Observation: If G is independent in X and $k := k(X)$ then G is (d,k)-*sparse*, i.e. $|E'| \leq d|V'| - k$ for all subgraphs (V', E'); if G is minimally rigid in X then G is (d,k)-*tight*, i.e. G is (d,k)-sparse and $|E| = d|V| - k$.

Proposition (Kitson and Power (2014))

Let $G = (V,E)$ be independent (resp. minimally rigid) in ℓ_q^d for $q \in [1,2) \cup (2,\infty]$. Then G is (d,d)-sparse (resp. (d,d)-tight).

A rigid example in ℓ_q^2 revisited

Define p to be the placement of the wheel graph W_5 with center v_0 in ℓ_q^2 where,

$$p_{v_0} = (0,0), \quad p_{v_1} = (-1,0), \quad p_{v_2} = (0,1), \quad p_{v_3} = (1,0), \quad p_{v_4} = (1,-1).$$

The altered rigidity matrix $\tilde{R}(W_5, p)$:

	$(v_0,1)$	$(v_0,2)$	$(v_1,1)$	$(v_1,2)$	$(v_2,1)$	$(v_2,2)$	$(v_3,1)$	$(v_3,2)$	$(v_4,1)$	$(v_4,2)$
v_0v_1	1	0	-1	0	0	0	0	0	0	0
v_0v_2	0	-1	0	0	0	1	0	0	0	0
v_0v_3	-1	0	0	0	0	0	1	0	0	0
v_0v_4	-1	1	0	0	0	0	0	0	1	-1
v_1v_2	0	0	-1	-1	1	1	0	0	0	0
v_2v_3	0	0	0	0	-1	1	1	-1	0	0
v_3v_4	0	0	0	0	0	0	0	1	0	-1
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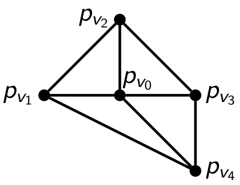


For $q = 2$, rank $\tilde{R}(W_5, p) = 2|V| - 3$, hence (W_5, p) is rigid in ℓ_2^2. For $q \neq 2$, rank $\tilde{R}(W_5, p) = |E| = 2|V| - 2$, hence (W_5, p) is minimally rigid in ℓ_q^2.

A flexible example in ℓ_q^2

Define p to be the placement of the triangle K_3 in ℓ_q^2 where for $0 < a < 1$ and $b > 0$,

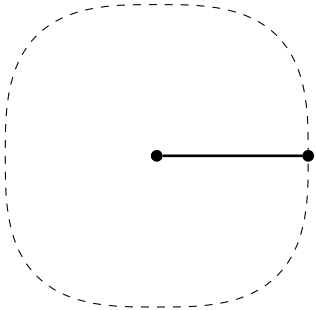
$$p_{v_1} = (0,0), \qquad p_{v_2} = (1,0), \qquad p_{v_3} = (a,b).$$

The altered rigidity matrix $\tilde{R}(K_3, p)$:

$$\begin{array}{c} \\ v_1v_2 \\ v_1v_3 \\ v_2v_3 \end{array} \begin{array}{c} \begin{array}{cccccc} (v_1,1) & (v_1,2) & (v_2,1) & (v_2,2) & (v_3,1) & (v_3,2) \end{array} \\ \left[\begin{array}{cccccc} -1 & 0 & 1 & 0 & 0 & 0 \\ -a^{q-1} & -b^{q-1} & 0 & 0 & a^{q-1} & b^{q-1} \\ 0 & 0 & (1-a)^{q-1} & -b^{q-1} & -(1-a)^{q-1} & b^{q-1} \end{array}\right] \end{array}$$

For any $q \in (1, \infty)$, rank $\tilde{R}(K_3, p) = |E| = 3$, hence (K_3, p) is independent in ℓ_q^2. As $|E| = 2|V| - 3$ then (K_3, p) is rigid in ℓ_q^2 if and only if $q = 2$.

A really weird flexible example in ℓ_q^2



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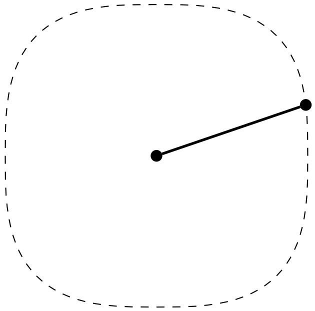

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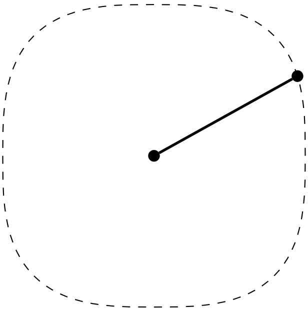

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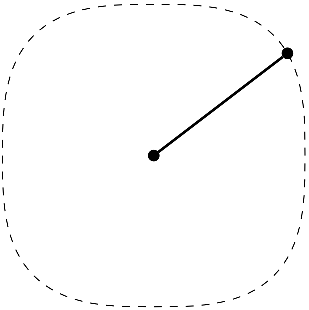

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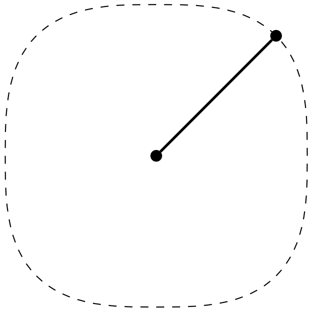

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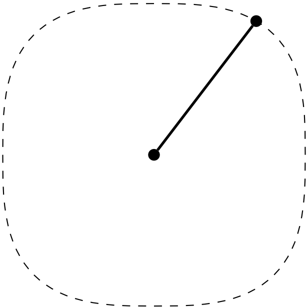

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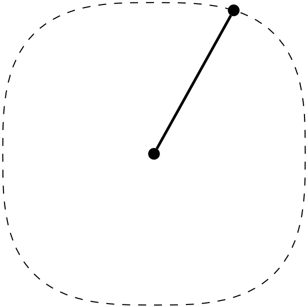

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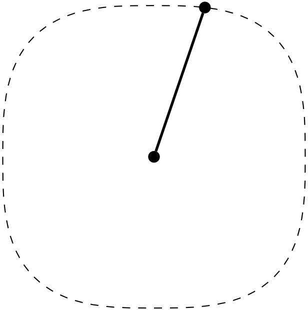

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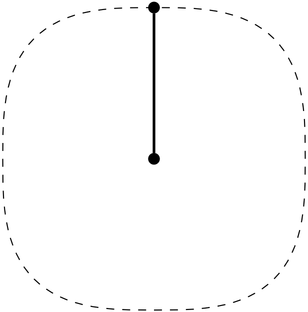

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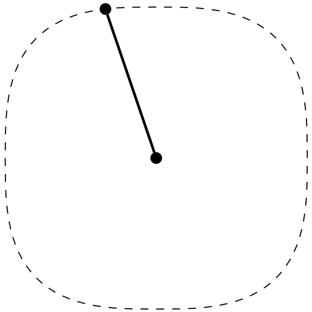

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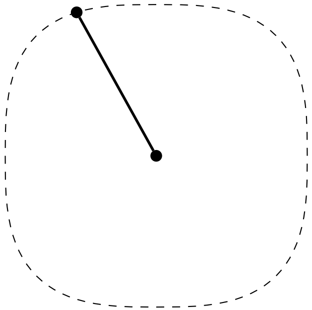

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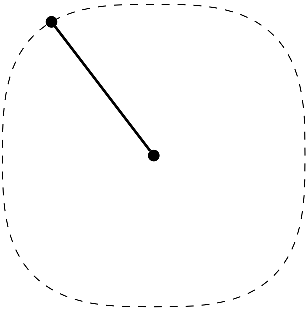

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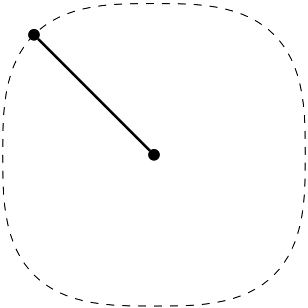

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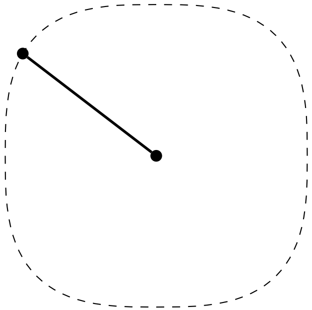

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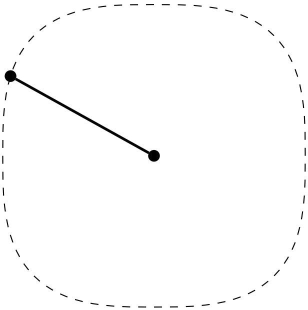

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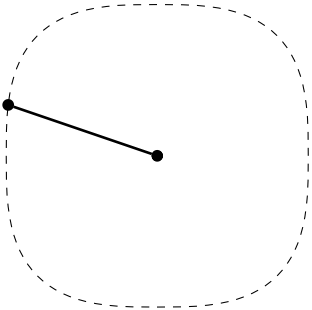

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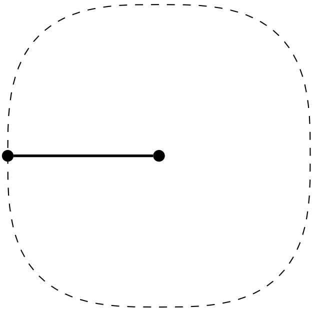

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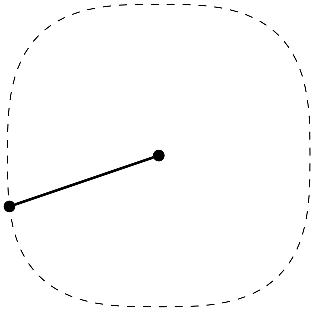

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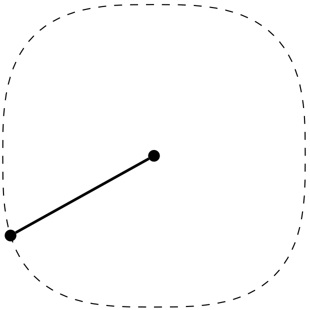

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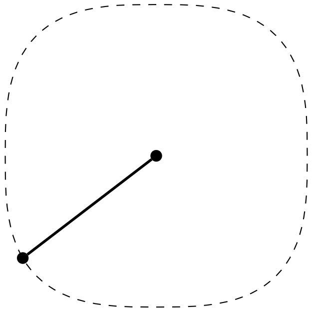

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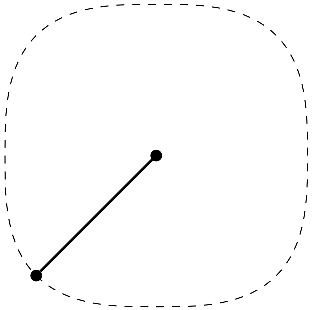

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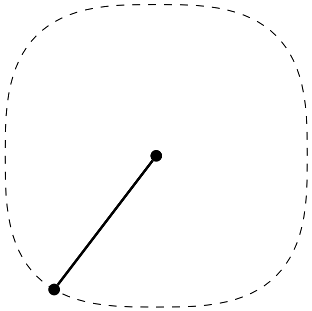

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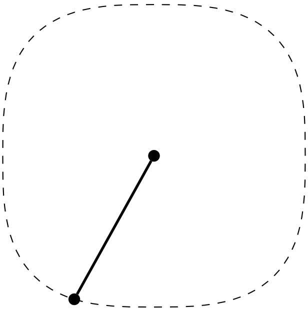

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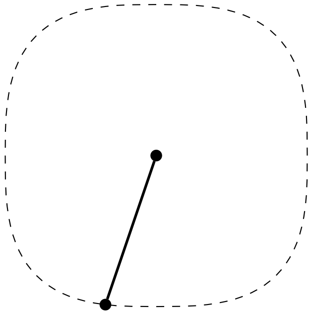

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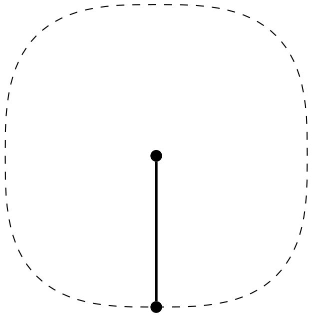

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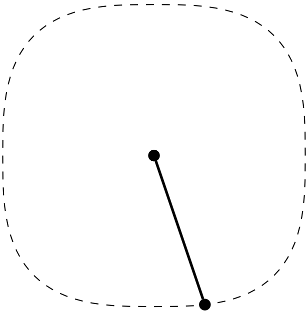

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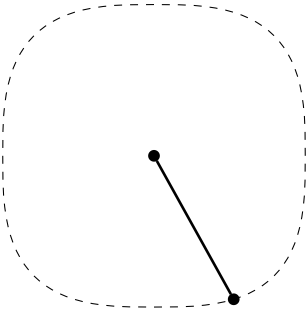

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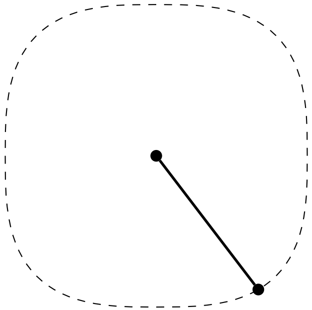

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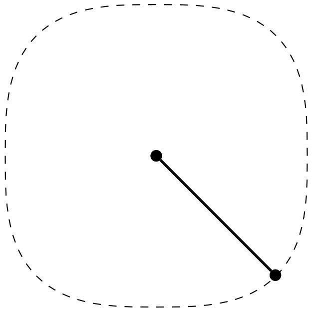

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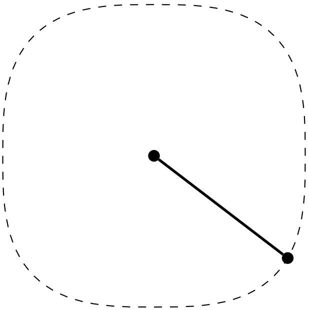

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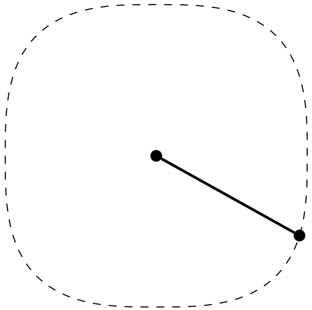

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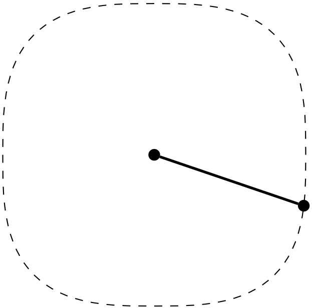

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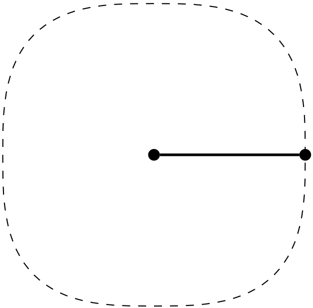


Rigidity in the plane

Theorem (Pollaczek-Geiringer (1927))

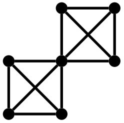
G is minimally rigid in ℓ_2^2 if and only if G is $(2,3)$-tight.

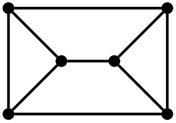
Theorem (Kitson and Power (2014))

For $q \in [1,2) \cup (2,\infty]$, G is minimally rigid in ℓ_q^2 if and only if G is $(2,2)$-tight.

Theorem (D. (2020))

If X is non-Euclidean, G is minimally rigid in X if and only if G is $(2,2)$-tight.





Strict convexity and smoothness

A normed space X with unit ball B is

(i) *strictly convex* if B is strictly convex, and
(ii) *smooth* if the boundary of B is a differentiable manifold.

Equivalently, a normed space X is

(i) *strictly convex* if $x \mapsto \varphi_x$ is injective on the set of smooth points, and
(ii) *smooth* if every non-zero point in X is smooth.

Importantly, X is strictly convex and smooth if and only if $x \mapsto \varphi_x$ is a homeomorphism from X to X^*.

Example

For $d \geq 2$, ℓ_q^d is strictly convex and smooth for $q \in (1, \infty)$, but neither for $q \in \{1, \infty\}$.

0- and 1-extensions

Figure: Examples a 3-dimensional 0-extension (left) and a 3-dimensional 1-extension (right).

- d-dimensional 0-extension: Add new vertex connected to d vertices.
- d-dimensional 1-extension: Split edge with new vertex and then connect new vertex to $d-1$ others.

Theorem

d-dimensional 0- and 1-extensions preserve independence (resp. rigidity) in strictly convex and smooth normed spaces.

Vertex splitting

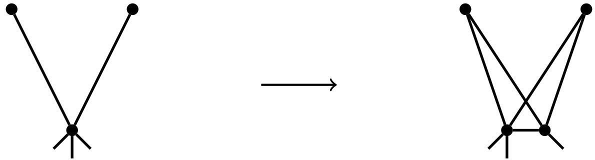


Figure: A 3-dimensional vertex split.

d-dimensional vertex split: Split vertex in two, attach $d-1$ neighbours to both copies, and then share the remaining edges out between the copies.

Theorem

d-dimensional vertex splitting preserves independence (resp. rigidity) in strictly convex and smooth normed spaces.

Graph substitution

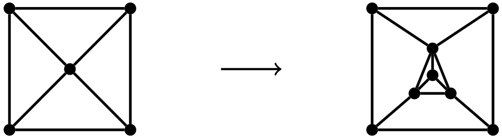


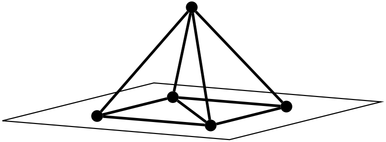
Figure: A vertex-to-K_4 substitution at the center vertex of W_5. This operation will preserve rigidity in any non-Euclidean 2-dimensional normed space.

Vertex-to-H substitution: Replace vertex with a copy of a graph H.

Theorem

Let X be a d-dimensional normed space and G' be a vertex-to-H substitution of G. If G and H are both independent in X then G' is independent in X. If $k(X) = d$ and H is minimally rigid, then G' is minimally rigid in X if and only if G is minimally rigid in X.

Coning operation

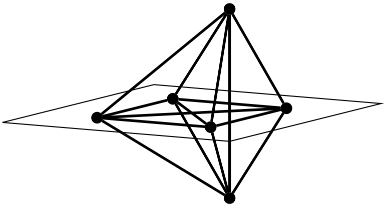


Coning: Add a new vertex connected to every vertex.

Theorem

Let G' be obtained from G by a coning operation. If G is independent in ℓ_q^d then G' is independent in ℓ_q^{d+1} for any $q \in (1, \infty)$. Furthermore, if G is minimally rigid in ℓ_q^d, then G' is minimally rigid in ℓ_q^{d+1} if and only if $q = 2$.

Bracing operation



k-vertex bracing: Add two vertices connected to k vertices and each other.

Theorem

Let G' be obtained from G by a $2d$-vertex bracing operation. If G is independent in ℓ_q^d then G' is independent in ℓ_q^{d+1} for any $q \in (1,2) \cup (2,\infty)$.

Corollary

The graph K_{2d} is minimally rigid in ℓ_q^d for all $q \in (1,2) \cup (2,\infty)$.

Important conjectures

Conjecture

The graph K_{2d} is rigid in every d-dimensional normed space.

The latter conjecture is known to be true for:

- Any smooth ℓ_q space.
- $\dim X = 2$; D. (2020).
- The 3-dimensional cylinder normed space; Kitson and Levene (2020).
- The 4-dimensional hypercyclinder normed space; Kitson and Levene (2020).

Open question: does every normed space have a rigid graph?

Conjecture

Let X be a normed space with $d = k(X) = \dim X$. Then G is minimally rigid in X if and only if G is (d, d)-tight.

Degree bounded graphs

Theorem

Let G be a connected graph with $\delta(G) \leq d+1$ and $\Delta(G) \leq d+2$ for any $d \geq 3$. Then for $q \in (1,2) \cup (2,\infty)$, the graph G is independent in ℓ_q^d if and only if G is (d,d)-sparse.

Theorem

Let X be a strictly convex and smooth 3-dimensional normed space. If $G = (V,E)$ is a graph where $|E'| \leq \frac{1}{2}(5|V'| - 7)$ for all subgraphs with at least one edge, then G is independent in X.

Triangulations of surfaces

For any triangulation $G = (V, E)$ of a compact surface S we have

$$3\chi(S) = 3|V| - |E|$$

where $\chi(S)$ is the *Euler characteristic* of S.

S	$\chi(S)$	$3\|V\| - \|E\|$
Sphere	2	6
Torus of genus g	2-2g	6(1-g)
Projective plane	1	3

Triangulations of the sphere

Theorem (Steinitz and Rademacher (1934))

Every triangulation of the sphere can be formed from K_4 by 3-dimensional vertex-splitting.

Theorem

Let X be a strictly convex and smooth 3-dimensional normed space. Then any triangulation of the sphere is independent in X.

Corollary

Let X be a strictly convex and smooth 3-dimensional normed space. Then any triangulation of the sphere is flexible in X.

Triangulations of the projective plane

Theorem (Barnette (1982))

Every triangulation of the projective plane can be formed from K_6 or $K_7 - K_3$ by 3-dimensional vertex-splitting.

Lemma

For $q \in (1,2) \cup (2,\infty)$, the graph $K_7 - K_3$ is minimally rigid in ℓ_q^3.

Theorem

For $q \in (1,2) \cup (2,\infty)$, any triangulation of the projective plane is minimally rigid in ℓ_q^3.

Thank you for listening!