Rigidity of linearly constrained frameworks in d-dimensions

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joint work with Bill Jackson (Queen Mary, London) and Shin-ichi Tanigawa (Tokyo)

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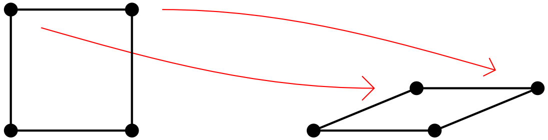
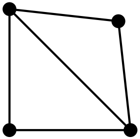
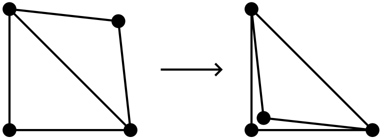
Rigidity

- A bar-joint framework (G, p) is the combination of a graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$.
- A framework (G, p) is (continuously) rigid if every edge-length preserving continuous motion of the vertices of (G, p) arises from an isometry of $\mathbb{R}^d$.





- This is rigid in 2D but has other realisations.

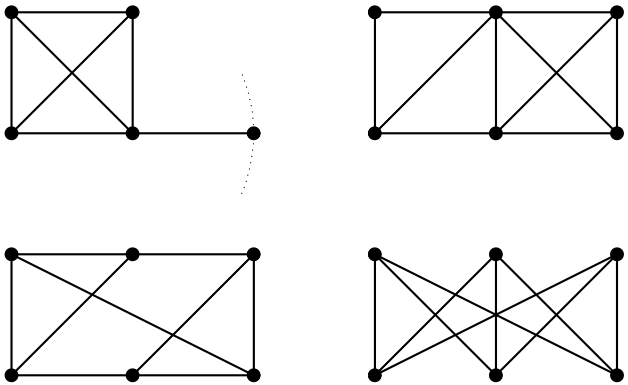


- A framework (G, p) is globally rigid if every framework (G, q) with the same edge lengths as (G, p) arises from an isometry of $\mathbb{R}^d$.
- This talk will focus on rigidity.

1D

- For frameworks on the real line, everything is simple:
- Folklore: A framework (G, p) is rigid in $\mathbb{R}$ if and only if G is connected.
- In dimension greater than 1 it is NP-hard to determine if a given framework is rigid (Abbott 2008).

Examples - in the plane



A linearisation

- An infinitesimal motion of a framework (G, p) is a map $\dot{p} : V \to \mathbb{R}^d$ such that $(p_j - p_i) \cdot (\dot{p}_j - \dot{p}_i) = 0$ for all $v_j v_i \in E$.
- The rigidity matrix is the $|E| \times d|V|$ matrix $R(G, p)$ whose rows are indexed by E and d-tuples of columns indexed by V in which, for $e = v_i v_j \in E$, the row has the form:

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- (G, p) is infinitesimally rigid if every infinitesimal motion is an infinitesimal isometry of $\mathbb{R}^d$, or equivalently if the rigidity matrix has rank $d|V| - \binom{d+1}{2}$.
- (G, p) is independent in $\mathbb{R}^d$ if $R(G, p)$ has linearly independent rows.

Asimow and Roth

- A framework (G, p) is generic if the coordinates of p form an algebraically independent set over $\mathbb{Q}$.

Theorem: Asimow and Roth 1978

Let (G, p) be a generic framework in $\mathbb{R}^d$. Then (G, p) is rigid if and only if it is infinitesimally rigid.

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- Hence, generically, rigidity is a property of the graph in every dimension.
- We say a graph G is d-rigid if some (and hence every) generic framework (G, p) in $\mathbb{R}^d$ is rigid.

Maxwell's necessary conditions

- A graph $G = (V, E)$ is $(d, \binom{d+1}{2})$-tight if $|E| = d|V| - \binom{d+1}{2}$ and for any subgraph (V', E'), with $|V'| \geq d$, we have $|E'| \leq d|V'| - \binom{d+1}{2}$.

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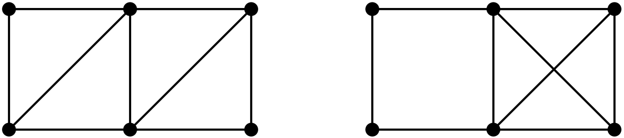
Lemma - Maxwell 1864

Let $G = (V, E)$ be d-rigid with $|V| \geq d + 1$. Then G contains a spanning subgraph H that is $(d, \binom{d+1}{2})$-tight.

- A major problem in rigidity theory is to establish sufficient combinatorial conditions for a graph to be d-rigid.

Laman's theorem

- A graph $G = (V, E)$ is (2, 3)-tight if $|E| = 2|V| - 3$ and for any subgraph (V', E') with $|V'| \geq 2$ we have $|E'| \leq 2|V'| - 3$.



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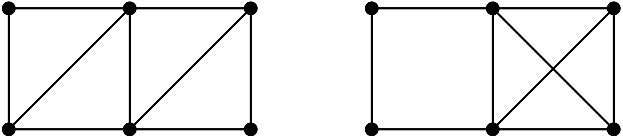
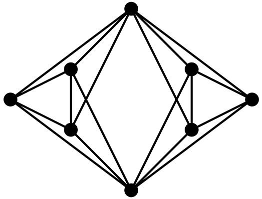


Theorem: Laman 1970, Pollaczek-Geiringer 1927

A graph G is 2-rigid if and only if G contains a spanning subgraph that is $(2, 3)$-tight.

- The converse fails in all dimensions $d \geq 3$.
- For example, here is a $(3, 6)$-tight graph that is flexible in $\mathbb{R}^3$.



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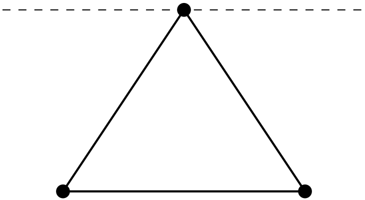

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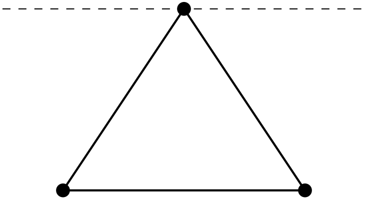
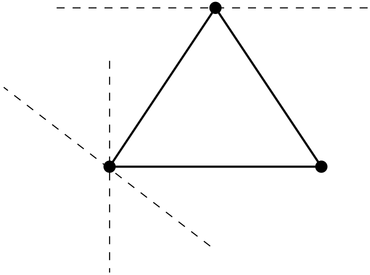


- If $e_i \in L$ is a loop at v_i, then q_i represents a normal vector to a hyperplane H containing p_i.
- Thus p_i is constrained to move only on the fixed hyperplane H.

- We can set up analogous definitions of rigidity for linearly constrained frameworks.



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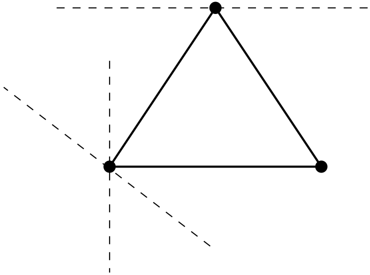

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- An infinitesimal motion of (G, p, q) is a map $\dot{p} : V \to \mathbb{R}^d$ satisfying the system of linear equations:

$$\begin{aligned} (p_i - p_j) \cdot (\dot{p}_i - \dot{p}_j) &= 0 \text{ for all } v_i v_j \in E \\ q_j \cdot \dot{p}_i &= 0 \text{ for all incident pairs } v_i \in V \text{ and } e_j \in L. \end{aligned}$$

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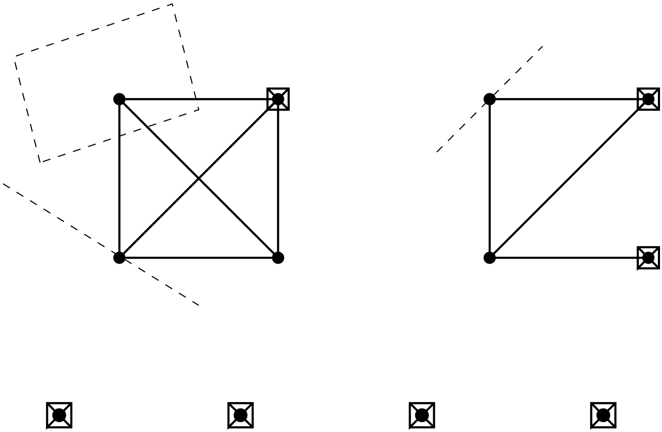
- The framework (G, p, q) is infinitesimally rigid if its only infinitesimal motion is $\dot{p} = 0$, or equivalently if rank $R(G, p, q) = d|V|$.
- We say G is independent in $\mathbb{R}^d$ if there exists a generic (G, p, q) in $\mathbb{R}^d$ such that $R(G, p, q)$ has linearly independent rows.

3D examples



Infinitesimal rigidity - 2D

Theorem - Streinu and Theran 2010

A generic linearly constrained framework (H, p, q) in $\mathbb{R}^2$ is (infinitesimally) rigid if and only if it has a spanning subgraph $G = (V, E, L)$ such that:

- $|E| + |L| = 2|V|$

and, for all subgraphs $G' = (V', E', L')$ of G we have

- $|E'| + |L'| \leq 2|V'|$ and
- $|E'| \leq 2|V'| - 3$ whenever $|E'| > 0$.

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- Related work:
  Servatius-Shai-Whiteley 2010,
  Katoh-Tanigawa 2013,
  Eftekhari-Jackson-N.-Schulze-Tanigawa-Whiteley 2019,
  Guler-Jackson-N 2020.

Infinitesimal rigidity - higher dimensions

- We will use $G^{[d-t]}$ to denote the graph formed from G by adding $d-t$ loops to each vertex.
- A graph $G=(V,E,L)$ is *t-sparse* if, for any subgraph (V',E',L'), we have $|E'|+|L'|\leq t|V'|$. Moreover it is is *t-tight* if $|E|+|L|=t|V|$ and it is t-sparse.

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Theorem: Cruickshank, Guler, Jackson, N. 2018

A generic linearly constrained framework $(G^{[d-t]},p,q)$ in $\mathbb{R}^d$, in which every vertex is constrained to an affine subspace of dimension $t\geq 1$ and $d\geq \max\{2t, t(t-1)\}$, is rigid if and only if G contains a spanning subgraph that is t-tight.

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- The complete graph K_5 is 2-tight but dependent in $\mathbb{R}^3$, and this generalises to K_{2t+1}.
- Hence $K_{2t+1}^{[d-t]}$ is not infinitesimally rigid in $\mathbb{R}^d$ when $d = 2t-1$.
- Therefore $d \geq 2t$ is in some sense 'necessary'.
- However by 'avoiding' K_{2t+1} we can deal with the case when $d = 2t-1$.

An improved bound

- A graph is H-free if it contains no subgraph isomorphic to H.

Theorem: Jackson-N-Tanigawa 2020+

Suppose $d \geq 2$ is an integer and $G = (V, E, L)$ is a looped simple graph with the property that every vertex of G is incident with at least $\lfloor \frac{d}{2} \rfloor$ loops. Then G is independent in $\mathbb{R}^d$ if and only if G is d-sparse and K_{d+2}-free.

- Note, a corresponding characterisation of rigidity follows quickly.
- I'll sketch the proof of the theorem in the rest of the talk.

Pinned frameworks

Lemma 1

Let $G = (V, E)$ be a simple graph with $P \subseteq V$ and $d \geq 2$ be an integer. Construct G' from G by adding d loops to each vertex of P and $\lfloor \frac{d}{2} \rfloor$ loops to each vertex of $V - P$. Suppose that G' is d-sparse and that G is K_{d+2}-free when d is odd. Then (G, P) is pinned independent in $\mathbb{R}^d$.

- Let H be the graph obtained from G' by deleting $\lfloor \frac{d}{2} \rfloor$ loops from every vertex.
- Then H is $\lceil \frac{d}{2} \rceil$-sparse and hence the minimum degree of H is at most $d+1$.
- Let v be a vertex of minimum degree in H and note that $v \in V - P$.
- We may now argue, using 0- and 1-extensions, that (G, P) is pinned independent.

Lemma 2

Let (G, p, q) be a generic linearly constrained framework in $\mathbb{R}^d$. Suppose that v is a vertex of G and $\operatorname{rank} R(G, p, q) = \operatorname{rank} R(G - \ell, p, q)$ for some loop ℓ incident to v. Then $\dot{p}(v) = 0$ for every infinitesimal motion $\dot{p}$ of (G, p, q).

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- We proceed by induction on $|E(G)|$.
- ℓ is contained in some d-circuit C.
- By induction we may suppose $G = C$ and that v is incident to at most $d - 1$ loops.
- Since G is a d-circuit, this implies that v is incident to an edge $e \in E(G)$.

- Let G^+ be obtained from G by adding a new loop ℓ^* at v and put $G^* = G^+ - \ell$.
- G and G^* are isomorphic graphs so are both d-circuits.
- By the circuit exchange axiom, there exists a d-circuit $G' \subseteq G^+ - e$.
- Since G and G^* are d-circuits, ℓ and ℓ^* are both loops in G'.
- Since $|E(G')| < |E(G)|$, we use induction to deduce that $\dot{p}(v) = 0$ for every infinitesimal motion of any generic (G', p', q') of G'.
- Since G' is a d-circuit, $\dot{p}(v) = 0$ for every infinitesimal motion of any generic realisation of $G' - \ell^*$.
- But $G' - \ell^* \subseteq G$, so this holds for G.

Theorem: Jackson-N-Tanigawa 2020+

Suppose $d \geq 2$ is an integer and $G = (V, E, L)$ is a looped simple graph with the property that every vertex of G is incident with at least $\lfloor \frac{d}{2} \rfloor$ loops. Then G is independent in $\mathbb{R}^d$ if and only if G is d-sparse and K_{d+2}-free.

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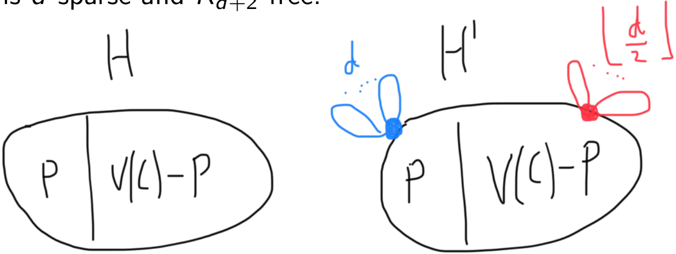
- Necessity is not difficult.
- Suppose G is d-sparse and K_{d+2}-free. We use induction on $|V| + |E|$.
- We may assume that $|V| \geq 2$ and G is connected.
- Suppose G has a d-tight subgraph H that is connected with $1 < |V(H)| < |V|$.

- Construct G' from G by replacing H with the d-tight K_{d+2}-free graph H' which has d loops at each vertex and no edges. By induction G is independent in $\mathbb{R}^d$.

- So we may assume G has no such subgraph.
- Assume, for a contradiction, that G is not independent in $\mathbb{R}^d$. Then G has a subgraph C which is a d-circuit.
- We use Lemma 2 to show that every vertex of C which is incident to a loop in C must be incident to d loops in G.

- Let $H = C - L(C)$ and P be the set of all vertices in C which are incident to at least one loop in C.
- Let H' be obtained from H by adding d loops at each vertex in P and $\lfloor \frac{d}{2} \rfloor$ loops at each vertex of $V(C) \setminus P$. Then H' is a subgraph of G so is d-sparse and K_{d+2}-free.



- We can now use Lemma 1 to deduce that (H, P) is pinned independent in $\mathbb{R}^d$.
- This contradicts the fact that C is an d-circuit.

Further work

- Improving the bound slightly might be possible by extending the idea of 'avoiding' K_{2t+1} to avoid multiple small (possibly flexible) circuits.
- Substantially improving the bound seems challenging.
- Other natural questions include:
  - global rigidity for linearly constrained frameworks, and
  - extending to symmetric (or other non-generic) linear constraints.

Thank you

Advert

- Thematic program - geometric constraint systems, framework rigidity and distance geometry, January - June 2021, `http://www.fields.utoronto.ca/activities/20-21/constraint`