Rigidity of linearly constrained frameworks in *d*-dimensions

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- A bar-joint framework (G, p) is the combination of a graph G = (V, E) and a map $p : V \to \mathbb{R}^d$.
- A framework (G, p) is (continuously) rigid if every edge-length preserving continuous motion of the vertices of (G, p) arises from an isometry of ℝ^d.





• This is rigid in 2D but has other realisations.



- A framework (G, p) is globally rigid if every framework (G, q) with the same edge lengths as (G, p) arises from an isometry of \mathbb{R}^d .
- This talk will focus on rigidity.

- For frameworks on the real line, everything is simple:
- Folklore: A framework (G, p) is rigid in \mathbb{R} if and only if G is connected.
- In dimension greater than 1 it is NP-hard to determine if a given framework is rigid (Abbott 2008).

Examples - in the plane



A linearisation

- An infinitesimal motion of a framework (G, p) is a map $\dot{p} : V \to \mathbb{R}^d$ such that $(p_j - p_i) \cdot (\dot{p}_j - \dot{p}_i) = 0$ for all $v_j v_i \in E$.
- The rigidity matrix is the |E| × d|V| matrix R(G, p) whose rows are indexed by E and d-tuples of columns indexed by V in which, for e = v_iv_i ∈ E, the row has the form:

$$(\ldots p_i - p_j \ldots p_j - p_i \ldots).$$

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- (G, p) is infinitesimally rigid if every infinitesimal motion is an infinitesimal isometry of \mathbb{R}^d , or equivalently if the rigidity matrix has rank $d|V| \binom{d+1}{2}$.
- (G, p) is independent in \mathbb{R}^d if R(G, p) has linearly independent rows.

Asimow and Roth

• A framework (G, p) is generic if the coordinates of p form an algebraically independent set over \mathbb{Q} .

Theorem: Asimow and Roth 1978

Let (G, p) be a generic framework in \mathbb{R}^d . Then (G, p) is rigid if and only if it is infinitesimally rigid.

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Let (G, p) be a generic framework in \mathbb{R}^d . Then (G, p) is rigid if and only if it is infinitesimally rigid.

- Hence, generically, rigidity is a property of the graph in every dimension.
- We say a graph G is d-rigid if some (and hence every) generic framework (G, p) in ℝ^d is rigid.

• A graph G = (V, E) is $\left(d, \binom{d+1}{2}\right)$ -tight if $|E| = d|V| - \binom{d+1}{2}$ and for any subgraph (V', E'), with $|V'| \ge d$, we have $|E'| \le d|V'| - \binom{d+1}{2}$.

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Lemma - Maxwell 1864

Let G = (V, E) be *d*-rigid with $|V| \ge d + 1$. Then *G* contains a spanning subgraph *H* that is $(d, \binom{d+1}{2})$ -tight.

• A major problem in rigidity theory is to establish sufficient combinatorial conditions for a graph to be *d*-rigid.

Laman's theorem

A graph G = (V, E) is (2,3)-tight if |E| = 2|V| - 3 and for any subgraph (V', E') with |V'| ≥ 2 we have |E'| ≤ 2|V'| - 3.



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Theorem: Laman 1970, Pollaczek-Geiringer 1927

A graph G is 2-rigid if and only if G contains a spanning subgraph that is (2,3)-tight.

- The converse fails in all dimensions $d \ge 3$.
- For example, here is a (3, 6)-tight graph that is flexible in \mathbb{R}^3 .



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- If e_i ∈ L is a loop at v_i, then q_i represents a normal vector to a hyperplane H containing p_i.
- Thus *p_i* is constrained to move only on the fixed hyperplane *H*.

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We say a linearly constrained framework (G, p, q) is generic if (p, q) is algebraically independent over Q.

 An infinitesimal motion of (G, p, q) is a map ṗ: V → ℝ^d satisfying the system of linear equations:

$$\begin{array}{rcl} (p_i-p_j)\cdot(\dot{p}_i-\dot{p}_j) &=& 0 \text{ for all } v_iv_j\in E \\ q_j\cdot\dot{p}_i &=& 0 \text{ for all incident pairs } v_i\in V \text{ and } e_j\in L. \end{array}$$

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The rigidity matrix R(G, p, q) is a (|E| + |L|) × d|V| matrix, in which: the row indexed by an edge v_iv_i ∈ E has the form:

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- The framework (G, p, q) is infinitesimally rigid if its only infinitesimal motion is $\dot{p} = 0$, or equivalently if rank R(G, p, q) = d|V|.
- We say G is independent in \mathbb{R}^d if there exists a generic (G, p, q) in \mathbb{R}^d such that R(G, p, q) has linearly independent rows.

3D examples



Theorem - Streinu and Theran 2010

A generic linearly constrained framework (H, p, q) in \mathbb{R}^2 is (infinitesimally) rigid if and only if it has a spanning subgraph G = (V, E, L) such that:

• |E| + |L| = 2|V|

and, for all subgraphs G' = (V', E', L') of G we have

 $\bullet \ |E'|+|L'|\leq 2|V'| \text{ and }$

•
$$|E'| \le 2|V'| - 3$$
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• Related work:

Servatius-Shai-Whiteley 2010,

Katoh-Tanigawa 2013,

Eftekhari-Jackson-N.-Schulze-Tanigawa-Whiteley 2019,

Guler-Jackson-N 2020.

Infinitesimal rigidity - higher dimensions

- We will use G^[d-t] to denote the graph formed from G by adding d-t loops to each vertex.
- A graph G = (V, E, L) is *t*-sparse if, for any subgraph (V', E', L'), we have $|E'| + |L'| \le t |V'|$. Moreover it is is *t*-tight if |E| + |L| = t |V| and it is *t*-sparse.

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Theorem: Cruickshank, Guler, Jackson, N. 2018

A generic linearly constrained framework $(G^{[d-t]}, p, q)$ in \mathbb{R}^d , in which every vertex is constrained to an affine subspace of dimension $t \ge 1$ and $d \ge \max\{2t, t(t-1)\}$, is rigid if and only if G contains a spanning subgraph that is t-tight.

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- We will improve the theorem by removing the assumption that $d \ge t(t-1)$.
- The complete graph K_5 is 2-tight but dependent in \mathbb{R}^3 , and this generalises to K_{2t+1} .
- Hence $K_{2t+1}^{[d-t]}$ is not infinitesimally rigid in \mathbb{R}^d when d = 2t 1.
- Therefore $d \ge 2t$ is in some sense 'necessary'.
- However by 'avoiding' K_{2t+1} we can deal with the case when d = 2t 1.

• A graph is H-free if it contains no subgraph isomorphic to H.

Theorem: Jackson-N-Tanigawa 2020+

Suppose $d \ge 2$ is an integer and G = (V, E, L) is a looped simple graph with the property that every vertex of G is incident with at least $\lfloor \frac{d}{2} \rfloor$ loops. Then G is independent in \mathbb{R}^d if and only if G is d-sparse and K_{d+2} -free.

- Note, a corresponding characterisation of rigidity follows quickly.
- I'll sketch the proof of the theorem in the rest of the talk.

Lemma 1

Let G = (V, E) be a simple graph with $P \subseteq V$ and $d \ge 2$ be an integer. Construct G' from G by adding d loops to each vertex of P and $\lfloor \frac{d}{2} \rfloor$ loops to each vertex of V - P. Suppose that G' is d-sparse and that Gis K_{d+2} -free when d is odd. Then (G, P) is pinned independent in \mathbb{R}^d .





- Let *H* be the graph obtained from *G'* by deleting $\lfloor \frac{d}{2} \rfloor$ loops from every vertex.
- Then H is ^d₂-sparse and hence the minimum degree of H is at most d + 1.
- Let v be a vertex of minimum degree in H and note that $v \in V P$.
- We may now argue, using 0- and 1-extensions, that (G, P) is pinned independent.

Lemma 2

Let (G, p, q) be a generic linearly constrained framework in \mathbb{R}^d . Suppose that v is a vertex of G and rank $R(G, p, q) = \operatorname{rank} R(G - \ell, p, q)$ for some loop ℓ incident to v. Then $\dot{p}(v) = 0$ for every infinitesimal motion \dot{p} of (G, p, q).

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- We proceed by induction on |E(G)|.
- ℓ is contained in some *d*-circuit *C*.
- By induction we may suppose G = C and that v is incident to at most d - 1 loops.
- Since G is a d-circuit, this implies that v is incident to an edge e ∈ E(G).



- Let G^+ be obtained from G by adding a new loop ℓ^* at v and put $G^* = G^+ \ell$.
- G and G^* are isomorphic graphs so are both d-circuits.
- By the circuit exchange axiom, there exists a *d*-circuit $G' \subseteq G^+ e$.
- Since G and G^* are d-circuits, ℓ and ℓ^* are both loops in G'.
- Since |E(G')| < |E(G)|, we use induction to deduce that p(v) = 0 for every infinitesimal motion of any generic (G', p', q') of G'.
- Since G' is a d-circuit, p(v) = 0 for every infinitesimal motion of any generic realisation of G' − ℓ*.
- But $G' \ell^* \subseteq G$, so this holds for G.

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- Necessity is not difficult.
- Suppose G is d-sparse and K_{d+2} -free. We use induction on |V| + |E|.
- We may assume that $|V| \ge 2$ and G is connected.
- Suppose G has a d-tight subgraph H that is connected with 1 < |V(H)| < |V|.

 Construct G' from G by replacing H with the d-tight K_{d+2}-free graph H' which has d loops at each vertex and no edges. By induction G is independent in ℝ^d.



- So we may assume G has no such subgraph.
- Assume, for a contradiction, that G is not independent in \mathbb{R}^d . Then G has a subgraph C which is a d-circuit.
- We use Lemma 2 to show that every vertex of *C* which is incident to a loop in *C* must be incident to *d* loops in *G*.

- Let H = C L(C) and P be the set of all vertices in C which are incident to at least one loop in C.
- Let H' be obtained from H by adding d loops at each vertex in P and L^d/₂ loops at each vertex of V(C) \ P. Then H' is a subgraph of G so is d-sparse and K_{d+2}-free.



- We can now use Lemma 1 to deduce that (H, P) is pinned independent in ℝ^d.
- This contradicts the fact that C is an d-circuit.

- Improving the bound slightly might be possible by extending the idea of 'avoiding' K_{2t+1} to avoid multiple small (possibly flexible) circuits.
- Substantially improving the bound seems challenging.
- Other natural questions include:
 - global rigidity for linearly constrained frameworks, and
 - extending to symmetric (or other non-generic) linear constraints.

Thank you

 Thematic program - geometric constraint systems, framework rigidity and distance geometry, January - June 2021, http: //www.fields.utoronto.ca/activities/20-21/constraint