Homothetic packings of centrally symmetric convex bodies

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Centrally symmetric convex bodies

A set $C \subset \mathbb{R}^d$ is a *convex body* if it is convex and compact with a non-empty interior. Given a convex body C that is *centrally symmetric* (i.e. if $x \in C$ then $-x \in C$), we can define a norm:

$$\|x\|_C := \inf\{\lambda > 0 : x \in \lambda C\}.$$

For a centrally symmetric convex body C:

- C is *smooth* if $\|\cdot\|_C$ is differentiable at every non-zero point. Example: rounded square.
- C is *strictly convex* if $\|\cdot\|_C$ is a strictly convex function. Example: intersection of two overlapping unit discs.
- C is a *regular symmetric body* if it is both smooth and strictly convex. Examples: discs, ℓ_p unit ball ($1 < p < \infty$).

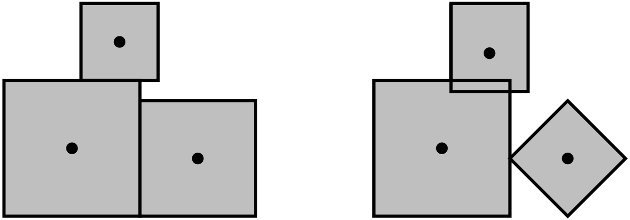
The *support* of x w.r.t. C is the vector $\varphi_C(x) \in \mathbb{R}^d$ that is the derivative of $\frac{1}{2}\|\cdot\|_C^2$ at x. We note $\varphi_C(0) = 0$ and $\varphi_C(\lambda x) = \lambda \varphi_C(x)$ for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^d$. If C is regular symmetric then the map $x \mapsto \varphi_C(x)$ is a well-defined homeomorphism from $\mathbb{R}^d$ to itself.

Homothetic packings

Let $C \subset \mathbb{R}^d$ be a centrally symmetric convex body. A *homothetic packing of* C, or C*-packing* for short, is a set $P := \{C_v : v \in V\}$ where

- for each $v \in V$, $C_v = r_v C + p_v$ for some $r_v > 0$ and $p_v \in \mathbb{R}^d$, and
- for each distinct pair $v, w \in V$, the interiors of C_v and C_w are disjoint.

Any C-packing is uniquely determined by its *placement* $p := (p_v)_{v \in V}$ and *radii* $r := (r_v)_{v \in V}$.

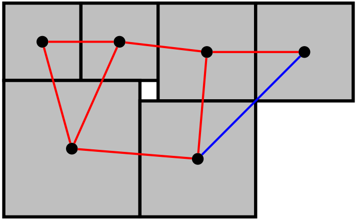


Contact graphs

We define the *contact graph of* P to be $G = (V, E)$, where $vw \in E$ if and only if $v \neq w$ and $C_v \cap C_w \neq \emptyset$. We note that for each distinct pair $v, w \in V$ we have

$$\|p_v - p_w\|_C \geq r_v + r_w,$$

with equality if and only if $vw \in E$.



If $P = (G, p, r)$ is a C-packing and $\| \cdot \|_C$ is differentiable at each point of $\{p_v - p_w : vw \in E\}$, then P is a *well-positioned* C-packing.

Flexes and stresses of packings

Let $C \subset \mathbb{R}^d$ be a centrally symmetric convex body and $P = (G, p, r)$ a C-packing.

- A *flexible motion* of P is a continuous path $\alpha : [0,1] \to \mathbb{R}^{d|V|}$ where
$$\|\alpha_v(t) - \alpha_w(t)\| \geq r_v + r_w$$
with equality if and only if $vw \in E$. If for each $t \in [0,1]$ there exists an isometry $f : \mathbb{R}^d \to \mathbb{R}^d$ with respect to $\| \cdot \|_C$ such that $\alpha(t) = f \circ p$, then α is *trivial*.
- Given P is well-positioned, an *infinitesimal flex* of P is a map $u : V \to \mathbb{R}^d$ where
$$\varphi_C(p_v - p_w).(u_v - u_w) = 0$$
for all $vw \in E$. If there exists an infinitesimal isometry $f : \mathbb{R}^d \to \mathbb{R}^d$ with respect to $\| \cdot \|_C$ such that $u = f \circ p$, then u is *trivial*
- Given P is well-positioned, an *equilibrium stress* of P is a map $a : E \to \mathbb{R}$ where
$$\sum_{w \in N(v)} a_{vw} \varphi_C(p_v - p_w) = 0$$
for all $v \in V$. We say a is *trivial* if $a_{vw} = 0$ for all $vw \in E$.

Sticky rigidity and independence

Let $C \subset \mathbb{R}^d$ be a centrally symmetric convex body and $P = (G, p, r)$ a C-packing.

- P is *sticky rigid* if all flexible motions of P are trivial; otherwise P is *sticky flexible*.
- Given P is well-positioned, P is *infinitesimally sticky rigid* if all infinitesimal flexes of P are trivial; otherwise P is *infinitesimally sticky flexible*.
- Given P is well-positioned, P is *independent* or *stress-free* if all equilibrium stresses of P are trivial; otherwise P is *dependent* or *stressed*.

Theorem (D. 2019)

If P is infinitesimally sticky rigid then it is sticky rigid. If P is independent and sticky rigid then it is infinitesimally sticky rigid.

Sparsity of generic circle packings

For $k, \ell \in \mathbb{N}$, graph $G = (V, E)$ is (k, ℓ)-*sparse* if $|E'| \leq k|V'| - \ell$ for all subgraphs $H = (V', E')$ with $E' \neq \emptyset$. If $|E| = k|V| - \ell$ also, then G is (k, ℓ)-*tight*.

Theorem (Connelly-Gortler-Theran 2019)

The following holds for any disc packing $P = (G, p, r)$ where r is generic (i.e. $\{r_v : v \in V\}$ is algebraically independent):

(i) *G is $(2, 3)$-sparse and P is independent.*

(ii) *G is $(2, 3)$-tight if and only if P is sticky rigid.*

The result will also hold for any regular symmetric body C that is the linear transform of a disc, with some alteration on the definition of generic.

Sparsity of almost all packings

Theorem

Let $C \subset \mathbb{R}^2$ be a regular symmetric body that is not the linear transform of a disc. Then for almost all radii $r \in \mathbb{R}^{|V|}_{>0}$, the following holds for any C-packing $P = (G, p, r)$:

(i) *G is $(2,2)$-sparse.*

(ii) *Given $k := 2|V| - |E| - 1$, if $\|\cdot\|_C$ is C^k-differentiable on $\mathbb{R}^2 \setminus \{0\}$ then P is independent.*

(iii) *G is $(2,2)$-tight if and only if P is infinitesimally sticky rigid.*

Sketch of proof

Let C be a regular symmetric body and $\|\cdot\|_C$ be C^k-differentiable.

- First, prove that every C-packing with contact graph G has no non-trivial *edge-length equilibrium stress*, i.e. an equilibrium stress $a : E \to \mathbb{R}$ where

$$\sum_{vw \in E} a_{vw} \|p_v - p_w\|_C = 0.$$

- Next, prove the set of all C-packings with contact graph G is a C^k-differentiable manifold of dimension $3|V| - |E|$.
- Now apply Sard's theorem to show that for almost all radii $r \in \mathbb{R}^{|V|}_{>0}$, the set $S_{G,C}(r)$ of all C-packings with contact graph G and radii r is either empty or a C^k-differentiable manifold of dimension $2|V| - |E|$ if $k \geq 2|V| - |E| - 1$.
- If we now choose r such that the above holds for all graphs G with vertex set V, then any C-packing with radii r cannot have contact graph where $|E| > 2|V| - 2$, since

$$2 \leq \dim S_{G,C}(r) = 2|V| - |E|.$$

Technicalities and limitations

- If C is not strictly convex then there exists a C-packing with a non-trivial edge-length equilibrium stress. It follows that the method cannot be used unless C is a regular symmetric body, as we require some smoothness for the manifolds we obtain.
- Sard's theorem limits our ability to determine whether a C-packing with random radii is independent; will be true if $k \geq 2|V| - |E| - 1$.

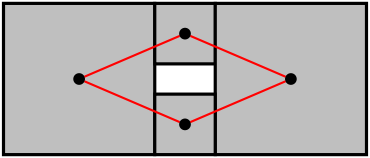


Figure: A homothetic square packing with an edge-length equilibrium stress.

Converse conjecture for generic circle packings

The following was originally conjectured by Connelly, Gortler and Theran.

Conjecture

Let G a $(2,3)$-sparse planar graph. Then there exists a disc packing with generic radii and contact graph G.

This is equivalent to:

Conjecture

Let G a $(2,3)$-sparse planar graph. Then there exists an independent disc packing with contact graph G.

Converse conjecture true for generic c. s. convex bodies

We denote by $\mathcal{K}_2$ the set of centrally symmetric convex bodies in $\mathbb{R}^2$, and we denote by $\mathcal{B}_2$ the set of regular symmetric bodies in $\mathbb{R}^2$. Their topologies are generated by the Hausdorff metric:

$$d_H(C, D) := \max \left\{ \sup_{x \in C} \inf_{y \in D} \|x - y\|, \sup_{x \in D} \inf_{y \in C} \|x - y\| \right\}.$$

Lemma

For each $(2,2)$-sparse planar graph G, there exists an open dense subset $\mathcal{G}_G \subset \mathcal{B}_2$ where the following holds; for every $C \in \mathcal{G}_G$, there exists an independent C-packing with contact graph G.

A subset is *comeagre* if it is the countable intersection of open dense sets.

- By the Baire Category Theorem, every comeagre subset of $\mathcal{K}_2$ is dense.
- $\mathcal{B}_2$ is a comeagre subset of $\mathcal{K}_2$ (Klee 59).

Theorem

There exists a comeagre subset $\mathcal{G} \subset \mathcal{K}_2$ where the following holds; for every $(2,2)$-sparse planar graph G and $C \in \mathcal{G}$, there exists an independent C-packing with contact graph G.

$\mathcal{G}_G$ is an open subset of $\mathcal{B}_2$

Fix $C' \in \mathcal{G}_G$ with independent C-packing (G, p', r'). We first show that there exists an open neighbourhood $U \subset \mathcal{B}_2$ of C' and a continuous map

$$f : U \to \mathbb{R}^{2|V|} \times \mathbb{R}^{|V|}_{>0}, \; C \mapsto (p, r),$$

such that $f(C') = (p', r')$ and if $f(C) = (p, r)$ then (G, p, r) is a C-packing.

We now note that the map

$$\mathcal{B}_2 \times \mathbb{R}^2 \to \mathbb{R}^2, \; (C, x) \mapsto \varphi_C(x)$$

is continuous. It now follows that for a sufficiently small open neighbourhood $U' \subset U$ of C', we have that the C-packing (G, p, r) is independent for each $C \in U'$ and $(p, r) = f(C)$.

$\mathcal{G}_G$ is a dense subset of $\mathcal{B}_2$ part 1: k-frames

Order the set V. For every graph $G = (V, E)$ and $\phi = (\phi_{v,w})_{vw \in E, v<w} \in \mathbb{R}^{d|E|}$, we say (G, ϕ) is *independent* if for every map $a : E \to \mathbb{R}$ where

$$\sum_{w \in N(v)} a_{vw} \phi_{v,w} = 0$$

for each $v \in V$ (we set $\phi_{w,v} = -\phi_{v,w}$), we have that $a_{vw} = 0$ for all $vw \in E$.

Theorem (White-Whiteley 87)

Let $X \subset \mathbb{R}^{2|E|}$ be the subset of elements ϕ where (G, ϕ) is independent. Then either:

(i) *G is (d, d)-sparse and X is an open dense subset of $\mathbb{R}^{2|E|}$, or*

(ii) *G is not (d, d)-sparse and $X = \emptyset$.*

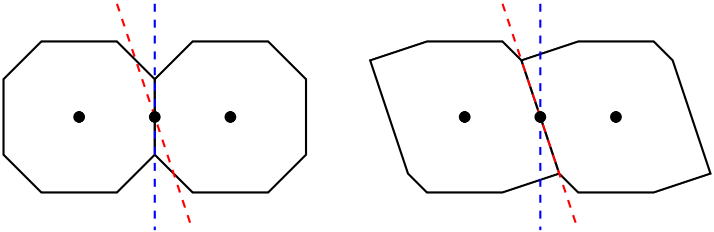
$\mathcal{G}_G$ is a dense subset of $\mathcal{B}_2$ part 2: slicing

Theorem (Schramm 90)

For every planar graph G and $C \in \mathcal{B}_2$ there exists a C-packing P with contact graph G.

Idea:

- Choose any $C \in \mathcal{B}_2$. By Schramm's result, there exists a C-packing $P = (G, p, r)$. Note: we can assume that for every pair $vw, v'w'$, the vectors $p_v - p_w$ and $p_{v'} - p_{w'}$ are linearly independent.
- By White and Whiteley's result, we can perturb the set $(\varphi_C(p_v - p_w))_{vw \in E}$ to find sufficiently close $\phi \in \mathbb{R}^{2|E|}$ where (G, ϕ) is independent.
- By altering C at each of the points $\pm(p_v - p_w)/\|p_v - p_w\|_C$, we can obtain sufficiently close C' to C with $\|p_v - p_w\|_{C'} = \|p_v - p_w\|_C$ and $\varphi_{C'}(p_v - p_w), \phi_{vw}$ linearly dependent for each $vw \in E$.
- Hence (G, p, r) is an independent C'-packing and $C' \in \mathcal{G}_G$.



Open questions

- Can we drop the requirement of central symmetry?
- Can similar methods be applied to packings where homotheticity is not required?
- It can be shown that any well-positioned homothetic square packing with random radii must also have a $(2, 2)$-sparse planar contact graph, though method does not extend to any other convex bodies. Is there a method of showing this is true for all convex polygons?
- If C is a regular symmetric body in $\mathbb{R}^2$ that isn't the linear transform of the disc and G is a $(2, 2)$-sparse planar graph, does there exist an independent C-packing with contact graph G? Important case: complete graph with four vertices.
- What can we say about packings in higher dimensions?

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