Characterizing the Universal Rigidity of Generic Tensegrities

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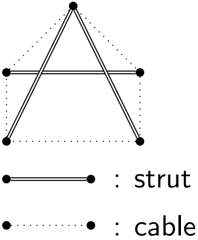
Brief Overview

- A tensegrity is a structure made from bars, cables and struts. A tensegrity is universally rigid if it is globally rigid in any dimension.
- Connelly(1982) showed that there is a compact certificate for universal rigidity.
- We showed that universally rigid generic tensegrities always have this certificate. We also extended this result to symmetric tensegrities.
- The proof relies on convex analysis and real representation theory.



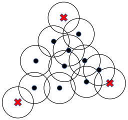
$$\begin{pmatrix} 3 & -4 & 2 & 0 & -1 \\ -4 & 6 & -4 & 2 & 0 \\ 2 & -4 & 4 & -4 & 2 \\ 0 & 2 & -4 & 6 & -4 \\ -1 & 0 & 2 & -4 & 3 \end{pmatrix}$$

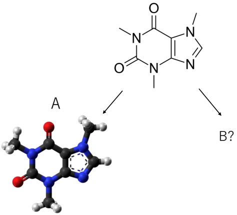
Certificate for UR

Why universal rigidity?

Motivation for global rigidity:

1. Sensor network localization
2. Structure analysis of molecules
3. Tensegrity construction





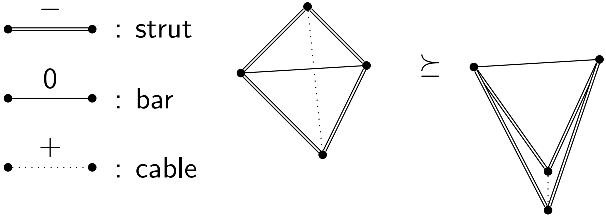
- Global rigidity $\rightarrow$ rank-constrained SDP
- Universal rigidity $\rightarrow$ SDP

Definition

- A *d-dimensional tensegirty* is a triple (G, σ, p) of
  - a finite graph G,
  - a sign map $\sigma : E(G) \to \{-, 0, +\}$,
  - a point configuration $p : V(G) \to \mathbb{R}^d$.

  When $\sigma(e) = 0 \quad (e \in E(G))$, it is called a framework.
- We denote $(G, \sigma, p) \succeq (G, \sigma, q)$ if

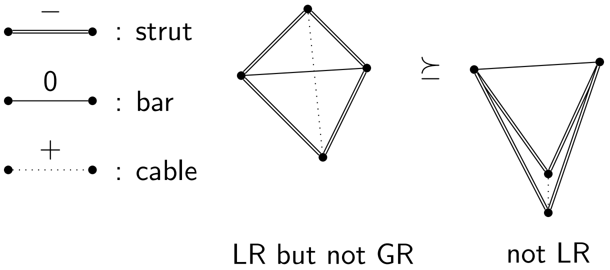
$$\|p_i - p_j\| \begin{array}{c} \leq \\ = \\ \geq \end{array} \|q_i - q_j\| \quad \begin{array}{l} (\sigma(ij) = -) \\ (\sigma(ij) = 0) \\ (\sigma(ij) = +) \end{array}$$



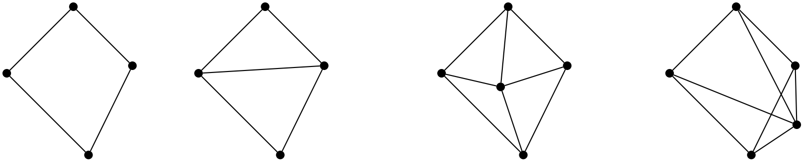
Definition

$$(G,\sigma,p) \succeq (G,\sigma,q) \iff \quad \|p_i - p_j\| \quad \begin{matrix} \le \\ = \\ \ge \end{matrix} \quad \|q_i - q_j\| \quad \begin{matrix} (\sigma(ij) = -) \\ (\sigma(ij) = 0) \\ (\sigma(ij) = +) \end{matrix}$$

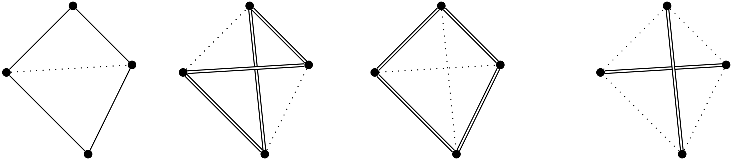
- (G,σ,p) is globally rigid if $(G,\sigma,p) \succeq (G,\sigma,q)$ implies that q is congruent to p.
- A tensegrity is *locally rigid* if it is globally rigid in its neighborhood.
- A tensegrity is universally rigid if it is globally rigid in any dimension. (Equivalently, locally rigid in any dimension.)



Examples



not LR | LR but not GR | GR but not UR | UR



not LR | not LR | LR but not GR | UR

- Universal rigidity is not a generic property.
- Rigidity of tensegirty is not a generic property.

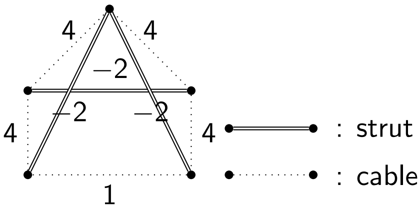
Equilibrium Stress

- $\omega : E(G) \to \mathbb{R}$ is a strictly proper equilibrium stress of a tensegirty (G, σ, p) if $\omega(e) > 0(\sigma(e) = +)$, $\omega(e) < 0(\sigma(e) = -)$ and

$$\sum_{j \in N_G(i)} \omega(ij)(p_j - p_i) = 0 \quad (i \in V(G)).$$

- For an edge weight $\omega : E(G) \to \mathbb{R}$, its weighted Laplacian $L_{G,\omega}$ is a $|V(G)| \times |V(G)|$ matrix defined by

$$L_{G,\omega} := \sum_{ij \in E(G)} \omega(ij)(\boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top.$$



$$L_{G,\omega} = \begin{pmatrix} 3 & -4 & 2 & 0 & -1 \\ -4 & 6 & -4 & 2 & 0 \\ 2 & -4 & 4 & -4 & 2 \\ 0 & 2 & -4 & 6 & -4 \\ -1 & 0 & 2 & -4 & 3 \end{pmatrix}$$

Super Stability

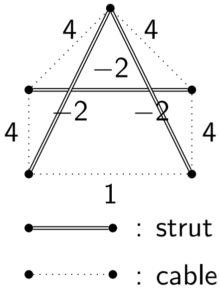
Conic condition

- Any affine image q of p satisfying $(G, \sigma, p) \succeq (G, \sigma, q)$ is congruent to p.

Theorem [Connelly 1982]

A d-dimensional tensegirty (G, σ, p) satisfying conic condition is universally rigid if there exists a strictly proper equilibrium stress $\omega : E(G) \to \mathbb{R}$ such that $\operatorname{rank} L_{G,\omega} = |V(G)| - d - 1$ and $L_{G,\omega} \succeq 0$.

_____ is called super stability.



$$L_{G,\omega} = \begin{pmatrix} 3 & -4 & 2 & 0 & -1 \\ -4 & 6 & -4 & 2 & 0 \\ 2 & -4 & 4 & -4 & 2 \\ 0 & 2 & -4 & 6 & -4 \\ -1 & 0 & 2 & -4 & 3 \end{pmatrix}$$

Eigenvalues: $14, 8, 0, 0, 0$

Theorem A

Super stability is sufficient for universal rigidty.

- A tensegity is generic if the coordinates of the point configuration are algebraically independent over $\mathbb{Q}$.

Theorem [Gortler-Thurston 2014]

For generic frameworks, super stability is necessary and sufficient for universal rigidity.

Theorem A

For generic tensegrities, super stability is necessary and sufficient for universal rigidity.

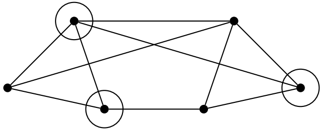
Symmetric Frameworks

Definition

Let Γ be a finite group and $\theta : \Gamma \to O(\mathbb{R}^d)$ be a group homomorphism. A d-dimensional framework (G, p) is θ-symmetric if

- Γ freely acts on $\mathrm{Aut}(G)$ and
- $\theta(\gamma)p_i = p_{\gamma i}$ $(\gamma \in \Gamma, i \in V(G))$.

A θ-symmetric framework is generic modulo symmetry if the coordinates of representative vertices are generic over $\mathbb{Q}_{\theta,\Gamma}$, which is a finite extension field of $\mathbb{Q}$ generated by entries of $\theta(\gamma)$ and representatives of real irreducible representations of Γ.



Theorem B

Definition

Let Γ be a finite group and $\theta : \Gamma \to O(\mathbb{R}^d)$ be a group homomorphism. A d-dimensional framework (G, p) is θ-symmetric if

- Γ freely acts on $\mathrm{Aut}(G)$ and
- $\theta(\gamma)p_i = p_{\gamma i}$ $(\gamma \in \Gamma, i \in V(G))$.

Theorem B

For any θ-symmetric framework which is generic modulo symmetry for some θ, super stability is necessary and sufficient for universal rigidity.

(The same statement holds for symmetric tensegrities.)

Position of Our Results

- [Connelly 2005] and [Gortler-Healy-Thurston 2010] showed that generic global rigidity is characterized by max-rank equilibrium stress matrix.
- [Connelly-Gortler-Theran 2020] showed that a graph is generically globally rigid if and only if it has a universally rigid generic realization.

	frameworks	tensegrities	with symmetry
GR	Connelly(2005) Gortler-Healy-Thurston(2010)		
UR	Connelly(1982)* Gortler-Thurston(2014)	Connelly(1982)* Theorem A	Connelly(1982)* Theorem B

Table: Algebraic characterization under genericity

$*$: Without genericity

- [Connelly and Gortler 2015] showed that the universal rigidity of tensegrities, not necessarily generic, is characterized by stronger condition than super stability.

Generic Framework Case: Step 1 [Gortler-Thurston 2014]

Let $n = |V(G)|$.

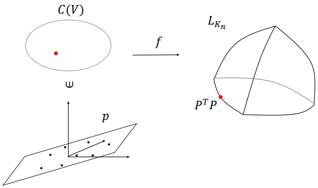
$$\begin{array}{ccc} \mathcal{C}(V) & \xrightarrow{f} & \mathcal{L}_{K_n} \\ & \searrow^{\pi \circ f} & \downarrow \pi \\ & & \mathcal{L}_G \end{array}$$

$$\mathcal{C}(V) := \{p : V(G) \to \mathbb{R}^n | \textstyle\sum_{i \in V(G)} p_i = 0\}$$

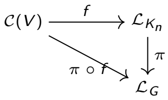
$$\mathcal{L}_G := \{L_{G,\omega} | \omega : E(G) \to \mathbb{R}\}$$

$$f(p) := P^\top P, \text{ where } P = \begin{pmatrix} p_1 & \cdots & p_n \end{pmatrix}.$$

$$\begin{array}{ccc} (\mathbb{R}^n)^d & \xrightarrow{q} & \mathbb{R}^{\binom{n}{2}} \\ & \searrow^{f_G} & \downarrow \pi \\ & & \mathbb{R}^e \end{array}$$



Generic Framework Case: Step 1 [Gortler-Thurston 2014]



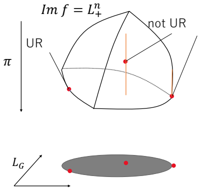
$$\mathcal{C}(V) := \{p : V(G) \to \mathbb{R}^n \mid \textstyle\sum_{i \in V(G)} p_i = 0\}$$

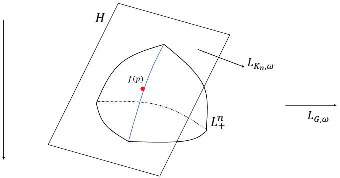
$$\mathcal{L}_G := \{L_{G,\omega} \mid \omega : E(G) \to \mathbb{R}\}$$

$$f(p) := P^\top P, \text{ where } P = \begin{pmatrix} p_1 & \cdots & p_n \end{pmatrix}.$$

Observation

- (G, p) is universally rigid $\iff \#(\pi^{-1}(\pi \circ f(p)) \cap \operatorname{Im} f) = 1$.
- $\operatorname{Im} f = \mathcal{L}_{K_n} \cap \mathcal{S}_+^n \simeq \mathcal{S}_+^{n-1}$.
- $L_{K_n,\omega} \in (\mathcal{L}_{K_n})^*$ exposes the smallest face containing $f(p) \iff \omega$ is an equilibrium stress on (K_n, p) and rank $L_{K_n,\omega} = n - d - 1$, $L_{K_n,\omega} \succeq 0$.

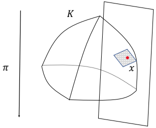




Generic Framework Case: Step 2 [Gortler-Thurston 2014]

Proposition [Gortler-Thurston 2014]

Let $K \subseteq \mathbb{R}^l$ be a closed, convex, line-free semialgebraic set and $\pi : \mathbb{R}^l \to \mathbb{R}^{l'}$ be a projection. A point $x \in K$ is locally generic within m-extreme points of K for some $m \in \mathbb{N}$ and $\#\pi^{-1}(\pi(x)) \cap K = 1$. Then, the smallest face containing x is exposed by a hyperplane parallel to π.



- By the genericity of p, $f(p)$ is generic in $\binom{d+1}{2}$-extreme points of Im f.
- The above proposition guarantees that we can take ω as an equilibrium stress on (G, p).

Proof of Theorem A

Theorem A

For generic tensegrities, super stability is necessary for universal rigidity.

$$\begin{array}{ccccc} \mathcal{C}(V) & \xrightarrow{\;f\;} & \mathcal{L}_{K_n} & \overset{\iota}{\hookrightarrow} & \mathcal{L}_{K_n} \times \mathbb{R}^{e\pm} \\ & {\scriptstyle \pi\circ f}\searrow & \downarrow {\scriptstyle \pi} & \swarrow {\scriptstyle \pi'} & \\ & & \mathcal{L}_G & & \end{array}$$

- $\pi'(L, s) := (L_{ij} + \sigma(ij)s_{ij})_{ij\in E}$.
- (G, σ, p) is universally rigid $\Leftrightarrow \#\pi'^{-1}(\pi'(f(p), \mathbf{0})) \cap (\mathrm{Im}\, f \times \mathbb{R}^{e\pm}_{\geq 0}) = 1$.
- We can prove local genericity of $(f(p), \mathbf{0})$ in $\binom{d+1}{2}$-extreme points of $\mathrm{Im}\, f \times \mathbb{R}^{e\pm}_{\geq 0}$.
- By Gortler-Thurston's propostion, super stability is guaranteed.

Proof of Theorem B: Step 1

Theorem B

For any θ-symmetric framework which is generic modulo symmetry for some θ, super stability is necessary and sufficient for universal rigidity.

Symmetric Laplacaians

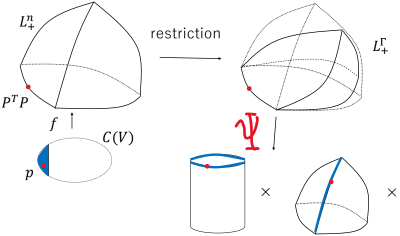
Let $\mathcal{L}_G^\Gamma := \{L_{G,\omega} | \omega : E(G) \to \mathbb{R}, \omega(\gamma i) = \omega(i)(i \in V(G), \gamma \in \Gamma)\}$.

$$\begin{array}{ccccc}
\mathcal{C}(V) & \xrightarrow{f} & \mathcal{L}_{K_n} & \xrightarrow{\text{proj}} & \mathcal{L}_{K_n}^\Gamma \\
& \searrow^{\pi \circ f} & \downarrow \pi & & \downarrow \pi|_{\mathcal{L}_{K_n}^\Gamma} \\
& & \mathcal{L}_G & \xrightarrow[\text{proj}]{} & \mathcal{L}_G^\Gamma
\end{array}$$

Proof of Theorem B: Step 2

$$
\begin{array}{ccccccc}
\mathcal{C}(V) & \xrightarrow{f} & \mathcal{L}_{K_n} & \xrightarrow{\text{proj}} & \mathcal{L}_{K_n}^{\Gamma} & \xrightarrow{\color{red}{\Psi}} & \color{red}{\bigoplus_{\rho\in\tilde{\Gamma}} I_{k_\rho}\otimes V_\rho} \\
& \searrow^{\pi\circ f} & \downarrow \pi & & \downarrow \pi|_{\mathcal{L}_{K_n}^{\Gamma}} & & \downarrow \Psi\circ\pi\circ\Psi^{-1} \\
& & \mathcal{L}_G & \xrightarrow[\text{proj}]{} & \mathcal{L}_G^{\Gamma} & \xrightarrow[\Psi|_{\mathcal{L}_G^{\Gamma}}]{} & \operatorname{Im}\Psi|_{\mathcal{L}_G^{\Gamma}}
\end{array}
$$

- Ψ is a decomposition of regular representation into real irreducible representations. $\tilde{\Gamma}$ is a equivalence set of real irreducible representations. V_ρ is a linear space defined by $\rho\in\tilde{\Gamma}$.



Proof of Theorem B: Step 3

Lemma

$\Psi \circ f(p)$ is locally generic in m-extreme points in $\operatorname{Im} \Psi \circ f$ for some $m \in \mathbb{N}$.

Proof Sketch.

1. Let $\mathcal{C}_\theta := \{p \in \mathcal{C}(V) : (G, p) \text{ is } \theta\text{-symmetric}\}$. $\underline{p \text{ is generic in } \mathcal{C}_\theta}$.
2. We can describe $\Psi \circ f(\mathcal{C}_\theta)$ by the orthogonality relation of real irreducible representations.
3. $\operatorname{Im} \Psi \circ f$ is isomorphic to the direct product of $\mathcal{S}^k_+$, $\mathcal{S}^k_{\mathbb{C},+}$ and $\mathcal{S}^k_{\mathbb{H},+}$.
4. 1, 2, 3 prove lemma.

□

$$
\begin{array}{ccccccc}
\mathcal{C}(V) & \xrightarrow{f} & \mathcal{L}_{K_n} & \xrightarrow{\text{proj}} & \mathcal{L}^\Gamma_{K_n} & \xrightarrow{\Psi} & \bigoplus_{\rho \in \tilde{\Gamma}} I_{k_\rho} \otimes V_\rho \\
& \searrow^{\pi \circ f} & \downarrow \pi & & \downarrow \pi|_{\mathcal{L}^\Gamma_{K_n}} & & \downarrow \Psi \circ \pi \circ \Psi^{-1} \\
& & \mathcal{L}_G & \xrightarrow[\text{proj}]{} & \mathcal{L}^\Gamma_G & \xrightarrow[\Psi]{} & \operatorname{Im} \Psi|_{\mathcal{L}_G}
\end{array}
$$

Concluding Remarks

	frameworks	tensegrities	with symmetry
GR	Connelly(2005)		
	Gortler-Healy-Thurston(2010)		
UR	Connelly(1982)*	Connelly(1982)*	Connelly(1982)*
	Gortler-Thurston(2014)	Theorem A	Theorem B

Table: Algebraic characterization under genericity