LMS Research School on Knowledge Exchange

Rigidity, Flexibility and Applications

Lancaster; 19-23 July 2021

`https://www.lancaster.ac.uk/maths/lms-ke-research-school-lancaster-2021/`

Lectures:

- Louis Theran (University of St Andrews) Geometric rigidity
- Miranda Holmes-Cerfon (NYU) Statistical mechanics and sphere packings
- Michael Farber (Queen Mary) Topology of linkages
- Silke Henkes (Bristol) Applications of rigidity to soft matter physics

Registration deadline: 31 January 2021

(Registration for research students is only 50 GBP).

Frameworks with coordinated constraints and a union of matroids

Bernd Schulze

Lancaster University

Joint work with Hattie Serocold and Louis Theran (St Andrews)

Outline

1. Brief introduction to rigidity
2. Coordinated edge motions: motivation and basic set-up
3. Generic coordinated rigidity
4. Further comments and open questions

Brief introduction to rigidity

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- (G, p) is called (locally) rigid if there exists a neighborhood of p in which every framework (G, q) that is equivalent to (G, p) is congruent to (G, p).
- (G, p) is called globally rigid if every framework (G, q) in $\mathbb{R}^d$ equivalent to (G, p) is congruent to (G, p).

Rigidity (cont.)

- A finite motion of (G, p) is a one-parameter family (G, p_t) with $p_0 = p$ and (G, p_t) equivalent to (G, p) for all $t \in [0, 1)$.

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- A finite motion is non-trivial if not all the (G, p_t) are congruent to (G, p).
- A framework is flexible if it has a non-trivial finite motion.
- **Theorem (Asimov-Roth, 78)**: Not rigid is equivalent to flexible.

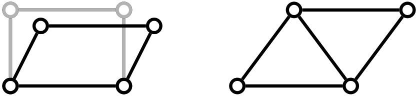

Figure: A flexible and a rigid framework in $\mathbb{R}^2$.

Infinitesimal rigidity

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- The dimension of the space of trivial infinitesimal motions of a framework (G, p) in $\mathbb{R}^d$ with $|V| \geq d$ is $\binom{d+1}{2}$. Thus, (G, p) is infinitesimally rigid if and only if $\mathrm{rank} R(p) = d|V| - \binom{d+1}{2}$.

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- A graph G is rigid (in dimension d) if some (any) generic realization of G in $\mathbb{R}^d$ is rigid.

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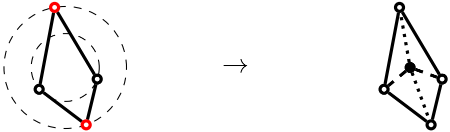
- There are well known combinatorial characterisations of isostatic or rigid graphs in $\mathbb{R}^2$ (e.g. Pollaczek-Geiringer, 1927, and Laman, 1970). Such a characterisation has not yet been found for higher dimensions.

Coordinated rigidity: motivation and basic set-up

Motivation

- Extension of work on frameworks on expanding spheres with independently variable radii (Nixon, S., Tanigawa and Whiteley, 2018).



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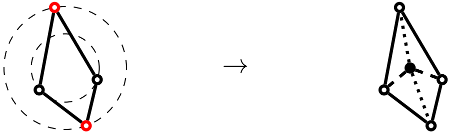

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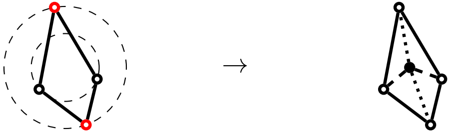

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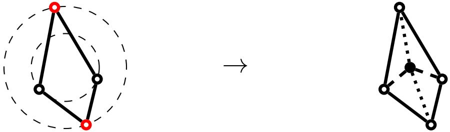

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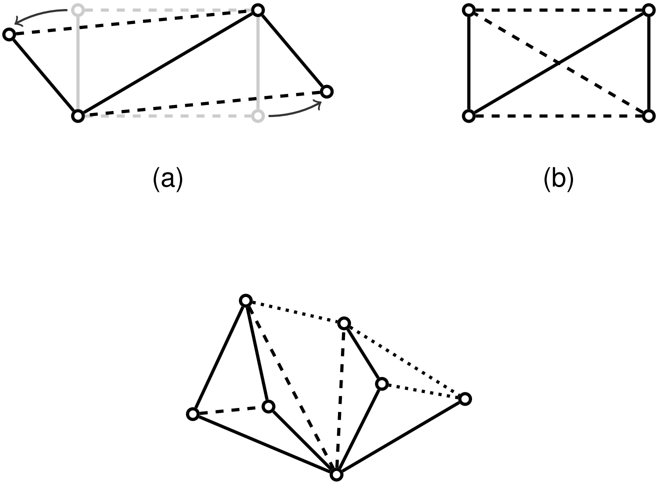
- Understanding structures appearing in engineering or materials science (e.g., mechanical linkages driven by coordinated pumps, or materials expanding at different rates when heated).
- Methodology for design of meta-materials.

Examples in 2D



(a) (b) (c)

Coordinated framework

- Let $k \in \mathbb{N}$, and $G = (V, E)$ be a graph. A coordination map is a function $c : E \to \{0, 1, \dots, k\}$. (G, c) is a k-coordinated graph.

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- A coordinated framework (G, c, p, r) is given by a k-coordinated graph (G, c) and a placement (p, r).
- A coordinated framework is generic if p is generic.

Equivalence and congruence

- (G, c, p, r) and (G, c, q, s) are equivalent if

$$\begin{aligned} ||p(j) - p(i)|| &= ||q(j) - q(i)|| && \text{for all } \{i, j\} \in E_0 \\ ||p(j) - p(i)|| + r(\ell) &= ||q(j) - q(i)|| + s(\ell) && \text{for all } \{i, j\} \in E_\ell, \text{ with } \ell \in [k] \end{aligned}$$

  and they are congruent if they are equivalent and p and q are congruent.

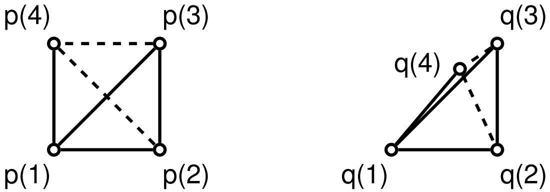

Figure: Two equivalent but non-congruent coordinated frameworks in the plane with $k = 1$.

Coordinated rigidity

- (G, c, p, r) is (locally) rigid if there is a neighborhood $U \subset \mathbb{R}^{|V|d+k}$ of (p, r) so that if $(q, s) \in U$ and (G, c, q, s) is equivalent to (G, c, p, r), then the two frameworks are congruent.

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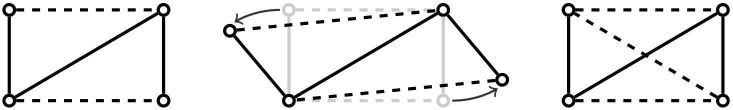
- A finite motion of (G, c, p, r) is a one-parameter family (G, c, p_t, r_t) with $(p_0, r_0) = (p, r)$ and all the (G, c, p_t, r_t) are equivalent to (G, c, p, r), for $t \in [0, 1)$.
  A finite motion is non-trivial if not all the (G, c, p_t, r_t) are congruent to (G, c, p, r). (G, c, p, r) is flexible if it has a non-trivial finite motion.



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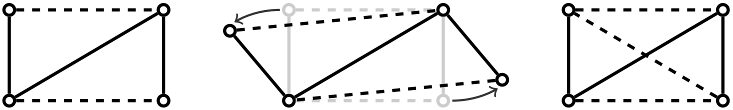

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- Not rigid is equivalent to flexible.

Coordinated infinitesimal rigidity

- By differentiating the constraints of a coordinated framework (G, c, p, r), we obtain the following linear system (with unknowns (p', r')):

$$\frac{[p(j) - p(i)] \cdot [p'(j) - p'(i)]}{\|p(j) - p(i)\|} = 0 \qquad \text{for all } \{i, j\} \in E_0 \qquad (1)$$

$$\frac{[p(j) - p(i)] \cdot [p'(j) - p'(i)]}{\|p(j) - p(i)\|} + r'(\ell) = 0 \quad \text{for all } \{i, j\} \in E_\ell, \text{ with } \ell \in [k] \quad (2)$$

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- **Lemma:** There is a non-zero solution to (1)–(2) if and only if there is a non-zero solution to:

$$[p(j)-p(i)]\cdot[p'(j)-p'(i)]=0 \qquad \text{for all } \{i,j\}\in E_0$$

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- An infinitesimal motion (p', r') of (G, c, p, r) is a velocity field p' supported on p and a vector $r' \in \mathbb{R}^k$ such that

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- Define $\mathbb{1}(c)$ to be the $|E| \times k$ matrix that has as its columns the characteristic vectors of the E_ℓ. Then the system above is equivalent to

$$R(p)p' + \mathbb{1}(c)r' = 0$$

where $R(p)$ is the rigidity matrix of the underlying framework (G, p).

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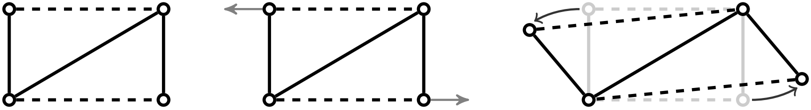
- The infinitesimal motions form a vector space that contains a $\binom{d+1}{2}$-dimensional subspace of motions $(p', \vec{0})$, with p' a trivial infinitesimal motion of (G, p).
- We define (G, p, c, r) to be infinitesimally rigid if these are the only infinitesimal motions, and infinitesimally flexible otherwise.

Infinitesimal vs finite rigidity

- Example of a nontrivial infinitesimal motion which extends to a finite motion:



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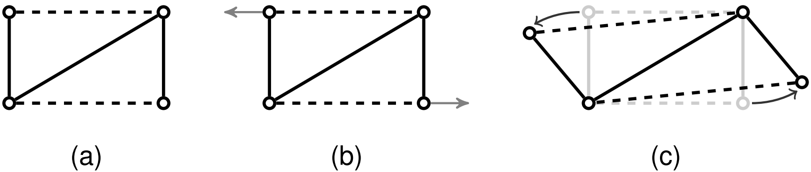

- Note: A finite motion preserves the edge-length differences between pairs of edges in the same coordination class.

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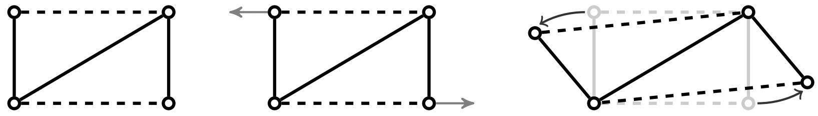
Infinitesimal vs finite rigidity

- Example of a nontrivial infinitesimal motion which extends to a finite motion:



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- **Theorem**: Infinitesimal and finite rigidity are equivalent for generic coordinated frameworks (by an Asimov-Roth type argument).

Generic coordinated rigidity (via redundant rigidity)

Main theorem

- Fix a dimension d and let $n \geq d$. Let E^n be the edges of K_n. The matroid $M_{d,n}$ on E^n of rank $dn - \binom{d+1}{2}$ that has as its bases the isostatic graphs with n vertices in $\mathbb{R}^d$ is called the d-dimensional rigidity matroid of K_n.

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- The restriction of $M_{d,n}$ to the edges of an n vertex graph G is the rigidity matroid of G, $M_d(G)$. $M_{d,n}$ is isomorphic to the linear matroid on the rows of the rigidity matrix $R(p)$ of K_n for any generic choice of p.

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- **Theorem (S., Serocold, Theran, 2019):** For $d \geq 1$ and $k \geq 1$, (G, c) is rigid in dimension d if and only if
  - G is rigid in dimension d and
  - some transversal $\{e_1, \ldots, e_k\}$ of the coordination classes $E_1, \ldots, E_k$ is redundant in $M_d(G)$.

Reformulation of main result via matroid unions

- Let E be a finite set and let $\mathcal{E} = \{E_1, \ldots, E_k\}$ be a collection of disjoint subsets of E. The transversal matroid $T_E(\mathcal{E})$ on E induced by the E_i has as its bases the sets

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- **Theorem:** Let (G, c) be a k-coordinated graph, and let $T_E(\mathcal{E})$ be the transversal matroid on E induced by the coordination classes $\mathcal{E} = \{E_1, \ldots, E_k\}$. Then the d-dimensional k-coordinated rigidity matroid of (G, c) is the union $M_d(G) \vee T_E(\mathcal{E})$.

Proof (necessity): key lemma

- To prove necessity, we need the following specialised fact from matroid theory (see, e.g., T. Brylawski, Constructions. in Theory of Matroids, N. White, editor, Cambridge UniversityPress, 1986).

- **Lemma:** Let M_1 and M_2 be two linearly representable matroids (over $\mathbb{R}$) on the same ground set. Then the matroid union $M_1 \vee M_2$ is also linearly representable, and a representation may be obtained by a matrix

$$(A, DB)$$

  where the rows of A represent M_1, the rows of B represent M_2, and D is a diagonal matrix of algebraically independent transcendentals.

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- Since the coordinated rigidity matrix of (G, c, p, r) has the form $R^+(p) = (R(p), \mathbb{1}(c))$, any independent set in the d-dimensional k-coordinated rigidity matroid of (G, c) must also be independent in $M_d(G) \vee T_E(\mathcal{E})$. So the rank of $R^+(p)$ is upper-bounded by

$$\max_{E' \subset E} \left\{ \mathrm{rank}_{M_d(G)} (E \setminus E') + \mathrm{rank}_{T_E(\mathcal{E})}(E') \right\} \leq d|V| - \binom{d+1}{2} + k. \quad (3)$$

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the framework (G', p) is infinitesimally rigid. This makes the edges in E' redundant. Since $\text{rank}_{T_E(\mathcal{E})}(E') = k$, E' is a transversal of $\mathcal{E}$.

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- Idea for latter step: make the edges $e_1, \ldots, e_k$ all very short. This makes the stress coefficients on them, relatively, large.

Proof (sufficiency): key lemmas

- **Lemma 1**: Suppose that (G, p) is a generic framework so that $(G \setminus e, p)$ is isostatic (and hence e is redundant). Let $e = \{i_1, i_2\}$ and define p^t to be like p except $p^t(i_2) = tp(i_2) + (1 - t)p(i_1)$. Let ω^t be the equilibrium stress of (G, p^t) with $\omega^t(e) = 1$. Then for all other edges f, we have for generic p_t,

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- **Lemma 2**: Suppose that (G, p) is a generic framework so that $(G \setminus e, p)$ is isostatic (and hence e is redundant). Then (G, p) has a unique equilibrium stress ω where $\omega(e) = 1$ and for all other edges f, we have
$$\omega(f) = (\det R^{i_1,\dots,i_d}_{e\to\times}(p))^{-1} \det(R^{i_1,\dots,i_d}_{f\to e}(p))$$
where the i_j are any tie-down vertices and $R^{i_1,\dots,i_d}_{f\to e}$ is obtained by removing the row corresponding to e and then replacing the row corresponding to f with it and $R^{i_1,\dots,i_d}_{e\to\times}$ by simply dropping the row corresponding to e.

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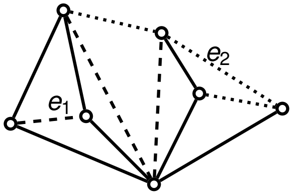
- Eventually we arrive at a configuration p^k which is generic and has $W\mathbb{1}(c)$ diagonally dominant.
- Thus, generically, the matrix

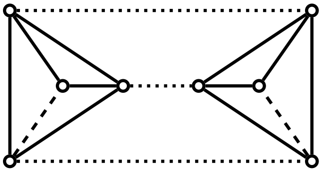
$$[R(p) \quad \mathbb{1}(c)]$$

has empty co-kernel, and hence rank $dn - \binom{d+1}{2} + k$.

Examples: d=k=2



(a)


(b)

Further comments and open questions

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  We have deterministic independence oracles for the matroids $M_{d,n}$ when $d = 1, 2$, and for $T_E(\mathcal{E})$.

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- Further work:
  - Extensions to other constraint systems (body-bar, direction-length, etc.)
  - Symmetric and periodic frameworks (partial results in Serocold's thesis)
  - Coordination classes maintaining sums or ratios of edge lengths
  - Global coordinated rigidity

Thank you!

Questions?

Reference:

- Bernd Schulze, Hattie Serocold and Louis Theran, *Frameworks with coordinated edge motions*, arXiv:1807.05376.