"Finiteness of fibers in matrix completion via Plücker coordinates"

1. Introduction

$M := M(r, m\times n)$ determinantal variety

$[i] := \{1, \dots, i\}$ $\quad K = \overline{K}$, $\dim M = r(m+n-r)$

$\Omega \subseteq [m] \times [n]$, $\pi_\Omega : M \longrightarrow K^{\#\Omega}$

algebraic matroid of M → Rosen+Sidman+Theran arXiv, 2019. "Matroids in action"

independent sets $=$ Ω's s.t. π_Ω is dominant

$\overset{\#\Omega = \dim M}{\Longleftrightarrow}$ $\exists$ non-empty open set $U \subseteq M$ s.t. $\pi_\Omega^{-1}(\pi_\Omega(X))$ is finite $\forall\, X \in U$.

goal: characterize the algebraic matroid

this talk: a (rich) family of basis sets

inspired by Pimentel / Alarcón+Boston+Nowak 2015, 2016 (CiS)

devices
i) connection between Plücker coordinates and matrix completion
ii) local coordinates on $Gr(r,m)$ induced by linkage matching ~~fields~~

Sturmfels+Zelevinsky, 1993

2. Plücker embedding/coordinates

$p : Gr(r,m) \hookrightarrow \mathbb{P}^N$, $N = \binom{m}{r} - 1$

$$\left.\begin{array}{l} S \in Gr(r,m) \\ B_S \in K^{m\times r} \end{array}\right\} \longmapsto \Big(\underbrace{(\psi)_S}_{\text{Plücker coordinates}} : \psi \subseteq [m],\ \#\psi = r \Big)$$

set of all $r\times r$ minors of B

3. Connection between matrix completion and Plücker coordinates

$$\Omega = \bigcup_{j\in[n]} \omega_j \times \{j\} \quad , \quad \omega_j \subseteq [m], \quad \omega_j \geq r$$

$$\mathcal{X} = [x_1 \dots x_n] \quad , \quad x_i \in K^m \qquad \pi_{\omega_j} : K^m \longrightarrow K^{\#\omega_j}$$

$$\pi_\Omega(\mathcal{X}) = \left[\pi_{\omega_1}(x_1) \dots \pi_{\omega_n}(x_n)\right]$$

$$\underbrace{\pi_{\omega_j}(x_j)}_{\text{Known}} \in \underbrace{\pi_{\omega_j}\big(\overset{\text{unKnown}}{\mathcal{B}(\mathcal{X})}\big)}_{\text{unKnown}} \longrightarrow \text{find } \mathcal{B}(\mathcal{X})$$

Suppose $\mathcal{B}(\mathcal{X})$ was known and $B = [b_1 \dots b_r]$ basis
If $\dim \pi_{\omega_j}(\mathcal{B}(\mathcal{X})) = r$ then

$$x_j = B\left(\pi_{\omega_j}(B)\right)^{\dagger} \pi_{\omega_j}(x_j) \quad \forall\, j$$

$\hookrightarrow$ can be formulated in Plücker coordinates

<u>Lem 1</u> $\varphi \subseteq [m]$, $\#\varphi = r+1$, $\dim \pi_\varphi(\mathcal{B}(\mathcal{X})) = r$

$x \in K^m$, $\pi_\varphi(x) \in \pi_\varphi(\mathcal{B}(\mathcal{X})) \Longleftrightarrow \sum (-1)^i x_i \left(\varphi \setminus \{\varphi_i\}\right)_{\mathcal{B}(\mathcal{X})} = 0$ ▨

$$\pi_\varphi(x) \perp \alpha = (\alpha_j)_{j\in[m]} \quad , \quad \alpha_{\varphi_i} = (-1)^i \left(\varphi \setminus \{\varphi_i\}\right), \; i \in [m-r]$$
$$\alpha_j = 0 \text{ otherwise}$$

This suggests working with a representation of $\mathcal{B}(x)^\perp$ in terms of $(\varphi)_{\mathcal{B}(\mathcal{X})}$

4. Standard local coordinates on $Gr(r,m)$

<u>Ex</u> $S = \mathcal{B}(\mathcal{X})$, $S \in Gr(2,6)$, $S^\perp \in Gr(4,6)$

$$B_{\mathcal{S}^\perp} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}, \quad (1234) \neq 0 \Rightarrow$$

$$B_{\mathcal{S}^\perp} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(2345)_{S^\perp} & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}$$

$(2345) = -\times$ $\qquad (2345)_{S^\perp} = (16)_S$

$$\Rightarrow B_{S^\perp} = \begin{bmatrix} (56)_S & 0 & 0 & 0 \\ 0 & (56)_S & 0 & 0 \\ 0 & 0 & (56)_S & 0 \\ 0 & 0 & 0 & (56)_S \\ -(16)_S & -(26)_S & -(36)_S & -(46)_S \\ (15)_S & (25)_S & (35)_S & (45)_S \end{bmatrix} \qquad \begin{array}{c} (1234)_{S^\perp} \neq 0 \\ \Longleftrightarrow \\ (56)_S \neq 0 \end{array}$$

<u>Rem 1</u> $\Phi = \bigcup_{j \in [4]} \varphi_j \times \{j\}$, $\quad \varphi_1 = \{1,5,6\}$, $\varphi_2 = \{2,5,6\}$, $\varphi_3 = \{3,5,6\}$, $\varphi_4 = \{4,5,6\}$

Let $x \in K^6$ and suppose that $\mathcal{V}_{\varphi_j}(x) \in \mathcal{V}_{\varphi_j}(\mathcal{S})$ $\forall j$ for some $\mathcal{S} \in Gr(2,6)$ with $(56)_{\mathcal{S}} \neq 0$. Then $x \in \mathcal{S}$. This follows directly from Lem 1.

5. Supports of Linkage Matching Fields

$\hookrightarrow$ Newton polytope of product of maximal minors

Sturmfels + Zelevinsky, 1993

$$\begin{array}{cc} & m-r \\ \Phi = m & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \end{array} \qquad \boxed{\# \bigcup_{j \in \mathcal{T}} \varphi_j \geq \# \mathcal{T} + r \quad \forall\, \mathcal{T} \subseteq [m-r]}$$

$m=6, \; r=2$

Dfn 1 SLMF of size $(m, m-r)$

$$\Phi = \bigcup_{j \in [m-r]} \varphi_j \times \{j\}, \quad \varphi_j \subseteq [m], \quad \# \varphi_j = r+1 \quad \square$$

Prp 1 (Pimentel-Alarcon+Boston+Nowak, 2015)

Let $\Phi = \bigcup_{j \in [m-r]} \varphi_j \times \{j\}$, $\varphi_j \subseteq [m]$, $\# \varphi_j = r+1$ be drawn uniformly at random. Then Φ is an SLMF$(m, m-r)$ with high-probability. $\square$

6. Local coordinates on $Gr(r,m)$ induced by SLMFs

Φ: SLMF of size $(m, m-r)$, $\mathcal{S} \in Gr(r,m)$

$$B_{\Phi,\mathcal{S}} = (b_{ij}) \leftarrow m \times (m-r)$$

$\varphi_{\ell j}$: ℓ-th element of φ_j

$$b_{ij} = (-1)^{\ell} \left(\varphi_j \setminus \{\varphi_{\ell j}\} \right)_{\mathcal{S}} \quad \text{if} \quad i = \varphi_{\ell j}$$

$$b_{ij} = 0, \quad \text{otherwise}$$

Prp 2 (Sturmfels+Zelevinsky, 1993)

Let Φ be SLMF of size $(m, m-r)$. Then $\exists$ hypersurface $\mathcal{V}_\Phi$ of $Gr(r,m)$ s.t. on $\mathcal{U}_\Phi = Gr(r,m) \setminus \mathcal{V}_\Phi$ the matrix $B_{\Phi,\mathcal{S}}$ is a basis for $\mathcal{S}^\perp$ $\forall\, \mathcal{S} \in \mathcal{U}_\Phi$. $\square$

7. Main Theorem

$\Omega = \bigcup_{j\in[n]} \omega_j \times \{j\}$, $\Omega_j' :=$ set of subsets of ω_j with $\#\omega_j = r+1$

Thm Suppose $\#\omega_j \geq r$ $\forall j$, $\#\Omega = \dim M$, $\exists$ partition $[n] = \bigcup_{\ell\in[r]} T_\ell$ s.t. $\forall\ \ell \in [r]$ $\exists\ \varphi_j^\ell \in \bigcup_{j'\in T_\ell} \Omega_{j'}'$ $\forall\ j \in [m-r]$ s.t.

$\Phi_\ell = \prod_{j\in[m-r]} \varphi_j^\ell \times \{j\}$ is SLMF of size $(m, m-r)$.

Then Ω is a basis set of the algebraic matroid of M. $\blacksquare$

8. Example

$M = M(2, 6\times 5)$, $r=2, m=6, n=5$

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad \omega_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Ω_1' consists of $\binom{5}{3}$ sets

$$\Phi_1 = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{cc} \underbrace{\quad\quad\quad\quad}_{\omega_1} & \underbrace{\quad}_{\omega_2} \end{array} \end{array}, \quad \Phi_2 = \begin{array}{c} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \uparrow\omega_3 & \underbrace{\quad\quad}_{\omega_4} & \uparrow\omega_5 \end{array} \end{array}$$

9. Proof Sketch

<u>Lem 1</u> $\Phi = \bigcup_{j \in [m-r]} \varphi_j \times \{j\}$: SLMF $(m, m-r)$

$\mathcal{S} \in \mathcal{V}_\Phi$, $x \in K^m$. If $\mathcal{M}_{\varphi_j}(x) \in \mathcal{M}_{\varphi_j}(\mathcal{S})$ $\forall j$ then $x \in \mathcal{S}$. ∎

$\mathcal{U}_1$ non-empty open of M where

$\mathcal{B}(\mathcal{X}) \in \bigcap_{\ell \in [r]} \mathcal{V}_{\Phi_\ell}$, $\dim \mathcal{M}_{\omega_j}(\mathcal{B}(\mathcal{X})) = r$

$\mathcal{M}_\Omega|_{\mathcal{U}_1} : \mathcal{U}_1 \longmapsto K^{\#\Omega}$, $\mathcal{X} \mapsto \mathcal{M}_\Omega(\mathcal{X})$

<u>Lem 2</u> $\exists\ \mathcal{X} \in \mathcal{U}$ s.t. $\mathcal{M}_\Omega|_{\mathcal{U}_1}^{-1}(\mathcal{M}_\Omega|_{\mathcal{U}_1}(\mathcal{X})) = \{\mathcal{X}\}$

<u>Prf</u> by example (§8)

$\mathcal{S} \in \bigcap_{\ell \in [r]} \mathcal{V}_{\Phi_\ell}$, $[b_1, b_2]$: basis of $\mathcal{S}$

$\mathcal{X} = [b_1\ b_1\ b_2\ b_2\ b_2]$ ∎

The rest is a consequence of the fiber dimension theorem. ∎