A Toroidal Maxwell-Cremona-Delaunay Correspondence

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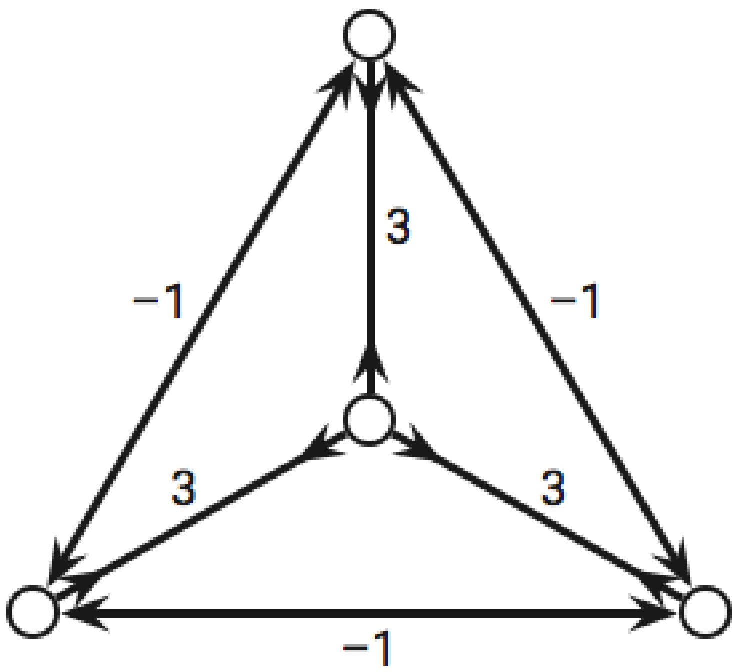
joint work with Jeff Erickson

MEET THE CAST

Equilibrium stress

[Maxwell 1864]

- Fix a straight-line plane graph G
- Assign a *stress* ω_e to every edge e
  - $\omega_e > 0 \Leftrightarrow$ e pulls inward
  - $\omega_e < 0 \Leftrightarrow$ e pushes outward



- ω is an *equilibrium stress* for G iff every vertex is a weighted average of its neighbors:

$$\sum_{v} \omega_{uv} \cdot (v - u) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Reciprocal Diagram

[Maxwell 1864]

Equilibrium stress for *G* ⇔ *reciprocal diagram G**

$$e^* \perp e$$
$$|e^*| = |\omega_e| \cdot |e|$$

- Faces of *G** certify equilibrium of *G*
- *G** may not be an *embedding*

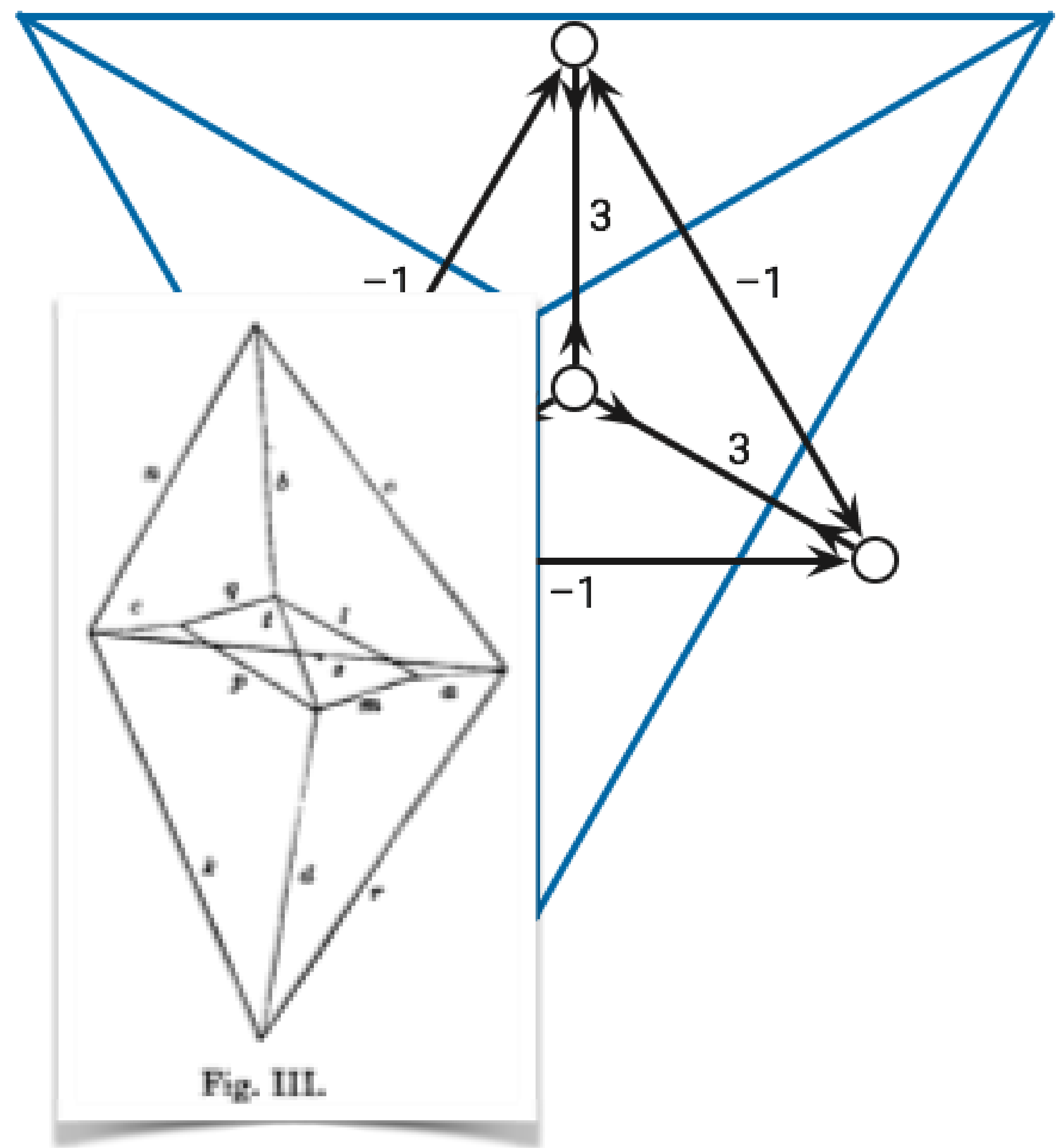


Fig. III.

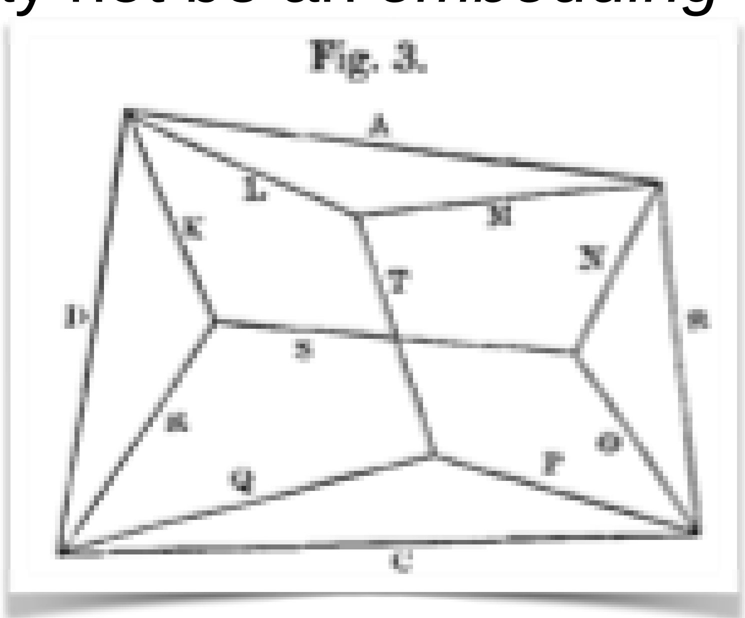


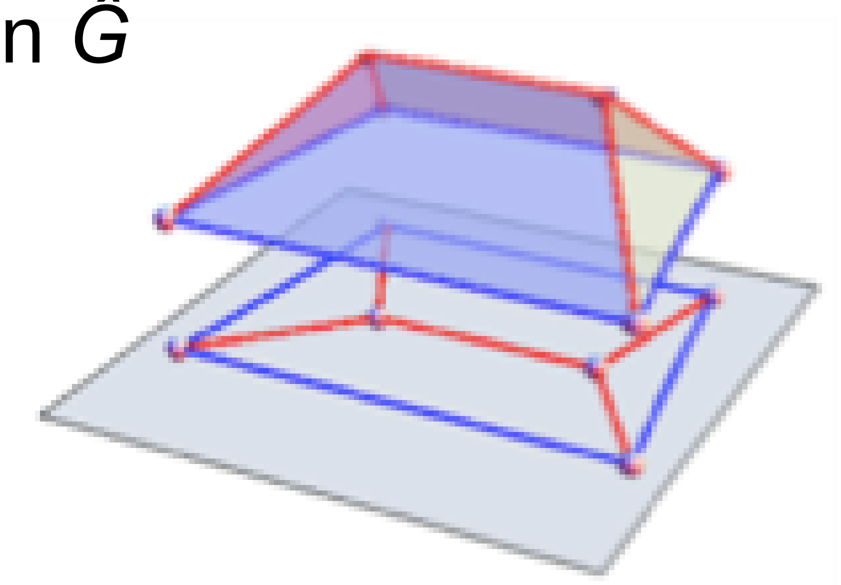
Fig. 3.

Polyhedral Lifting

[Maxwell 1864]

Equilibrium stress for G ⇔ *polyhedral lifting* $\hat{G}$

- $\hat{G}$ is a straight-line graph in 3-space, not all in one plane
  - G is the orthogonal projection of $\hat{G}$
  - Every face of G lifts to a planar polygon in $\hat{G}$
- For any *interior* edge *e:*
  - $\hat{e}$ is *convex* ⇔ $\omega_e > 0$
  - $\hat{e}$ is *concave* ⇔ $\omega_e < 0$



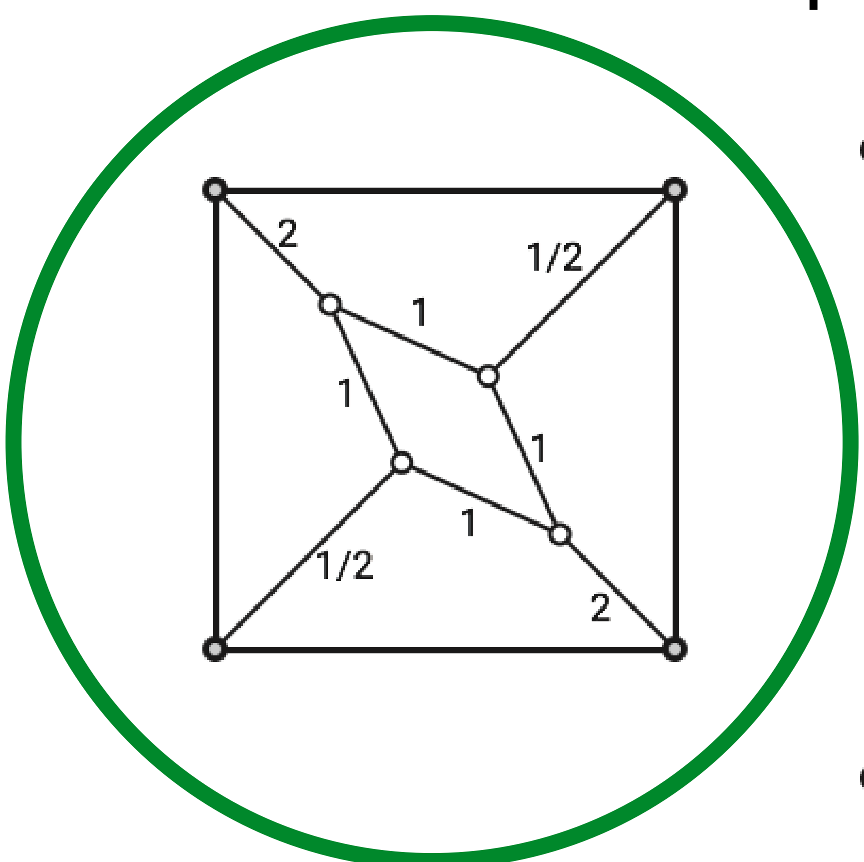
[Borcea Streinu 2015]

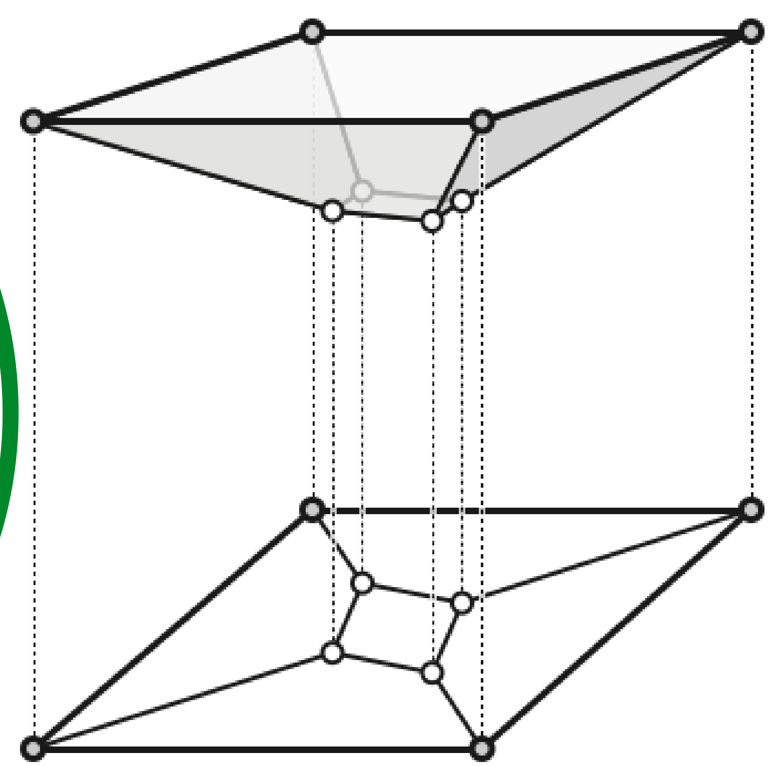
Positive Equilibrium

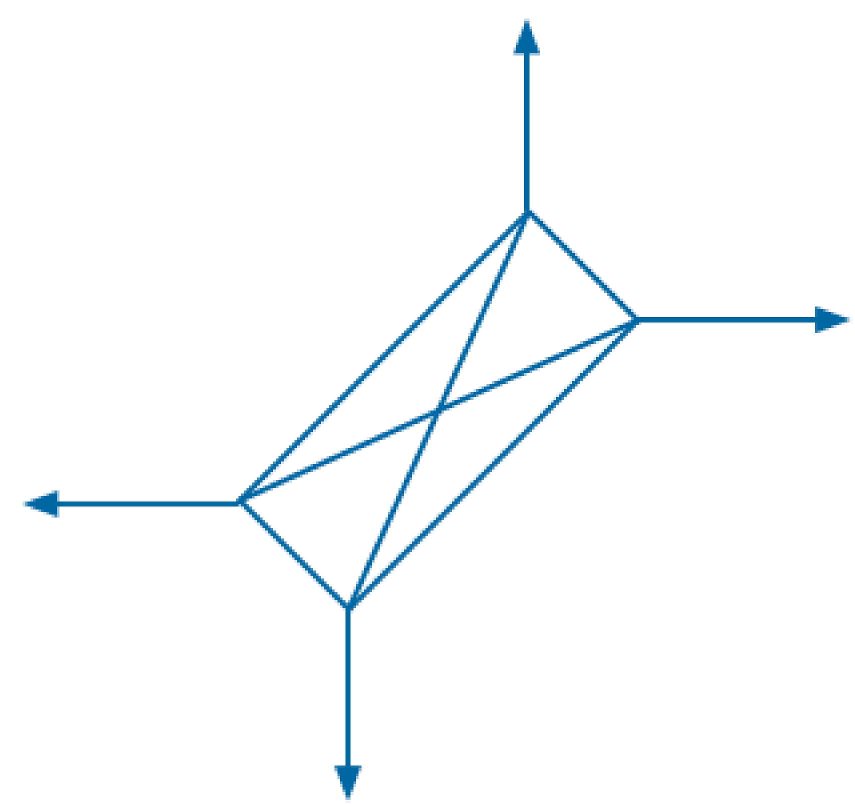
[Maxwell 1864]

If the outer face of G is convex and $\omega_e > 0$ for every *interior* edge e, then the following are equivalent:

- *Interior* equilibrium stress ω for G
- *Convex* polyhedral lift $\hat{G}$
- *Embedded* reciprocal diagram G^*



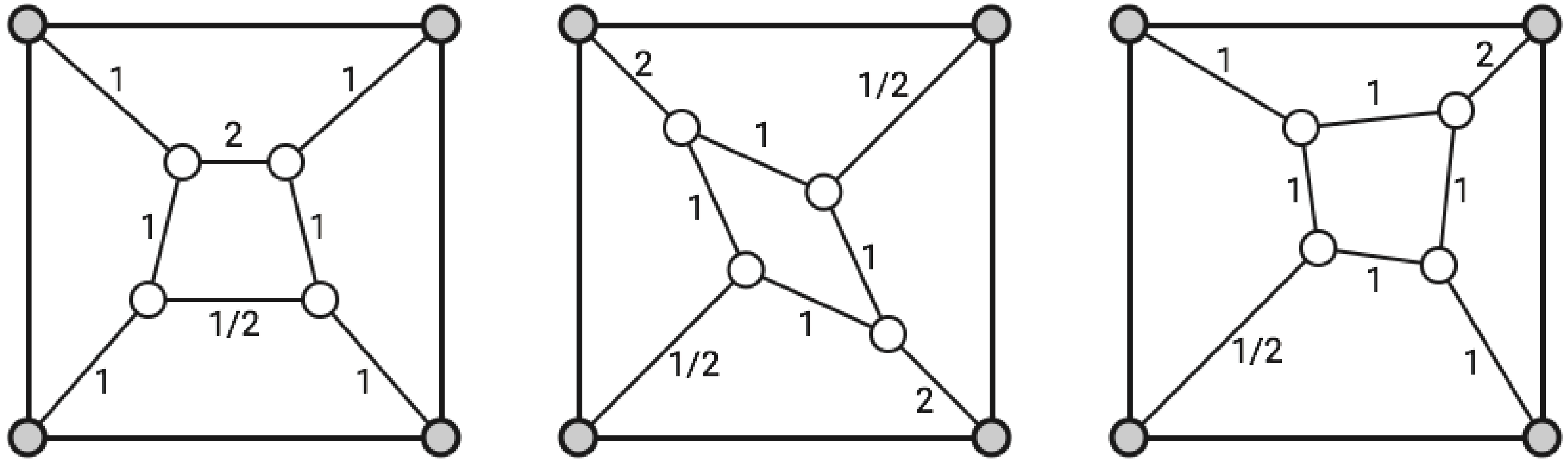




Tutte's spring embedding

[Tutte 1963]

- Suppose *G* is 3-connected and outer face of *G* is convex
- Assign arbitrary *positive* stresses $\omega_e > 0$ to *interior* edges *e*
- Minimize energy function $\Phi(G,\omega) := \sum_e \omega_e \cdot |e|^2$
  - Solve the Laplacian linear system $\sum_v \omega_{uv} \cdot (v - u) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- Straight-line *embedding* of *G* with interior equilibrium stress ω ⇒ convex lift ⇒ convex faces



Positive Equilibrium

[Maxwell 1864]

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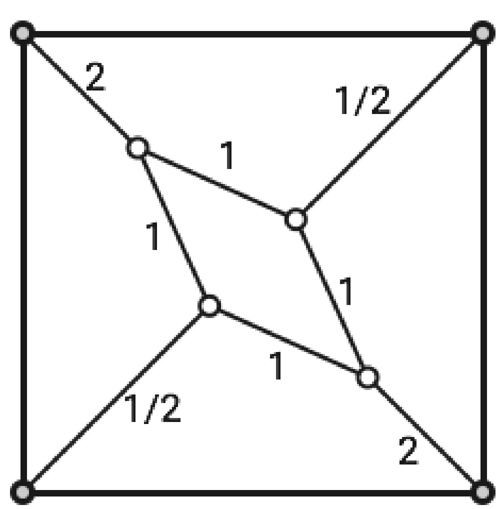
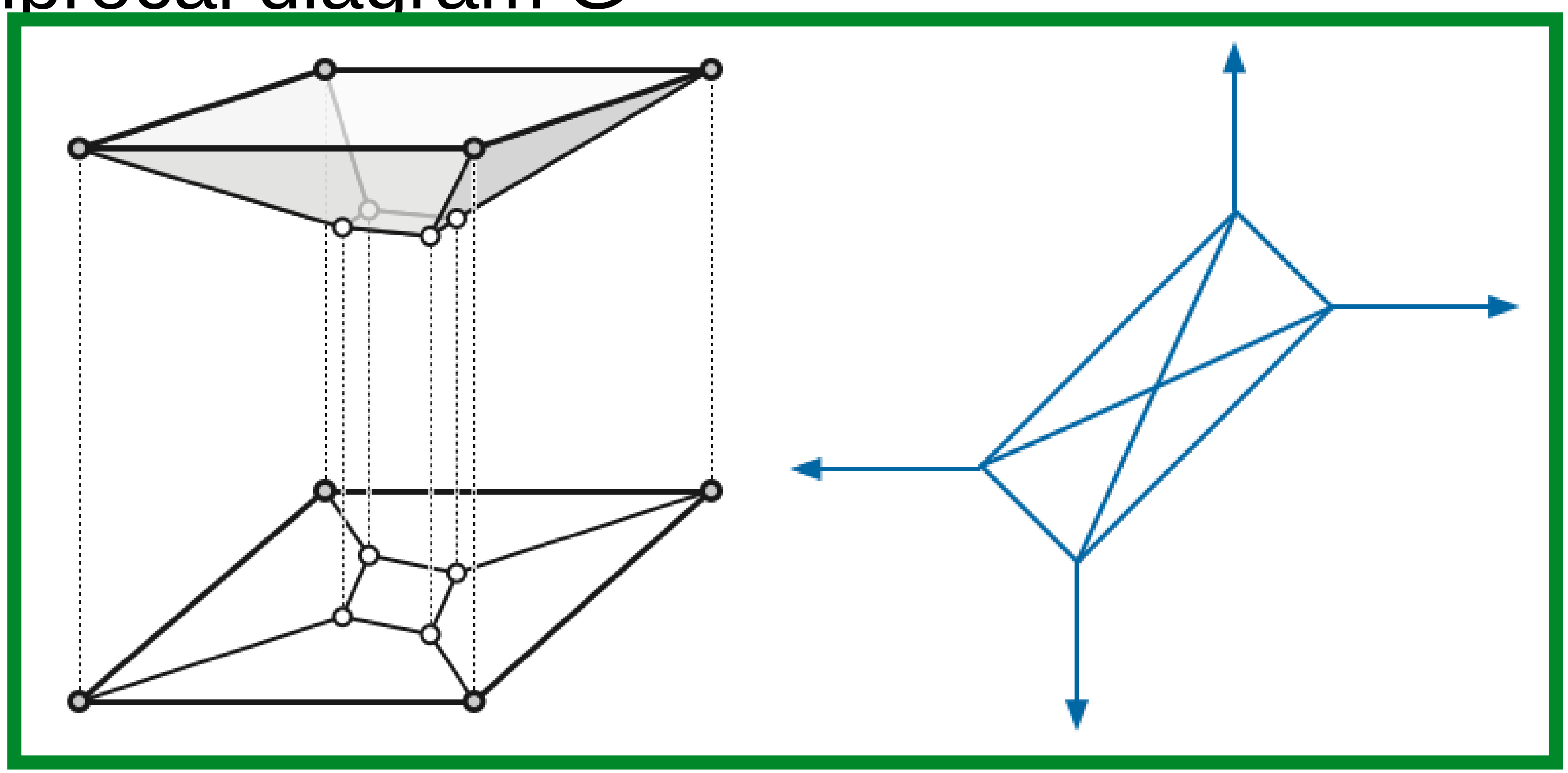
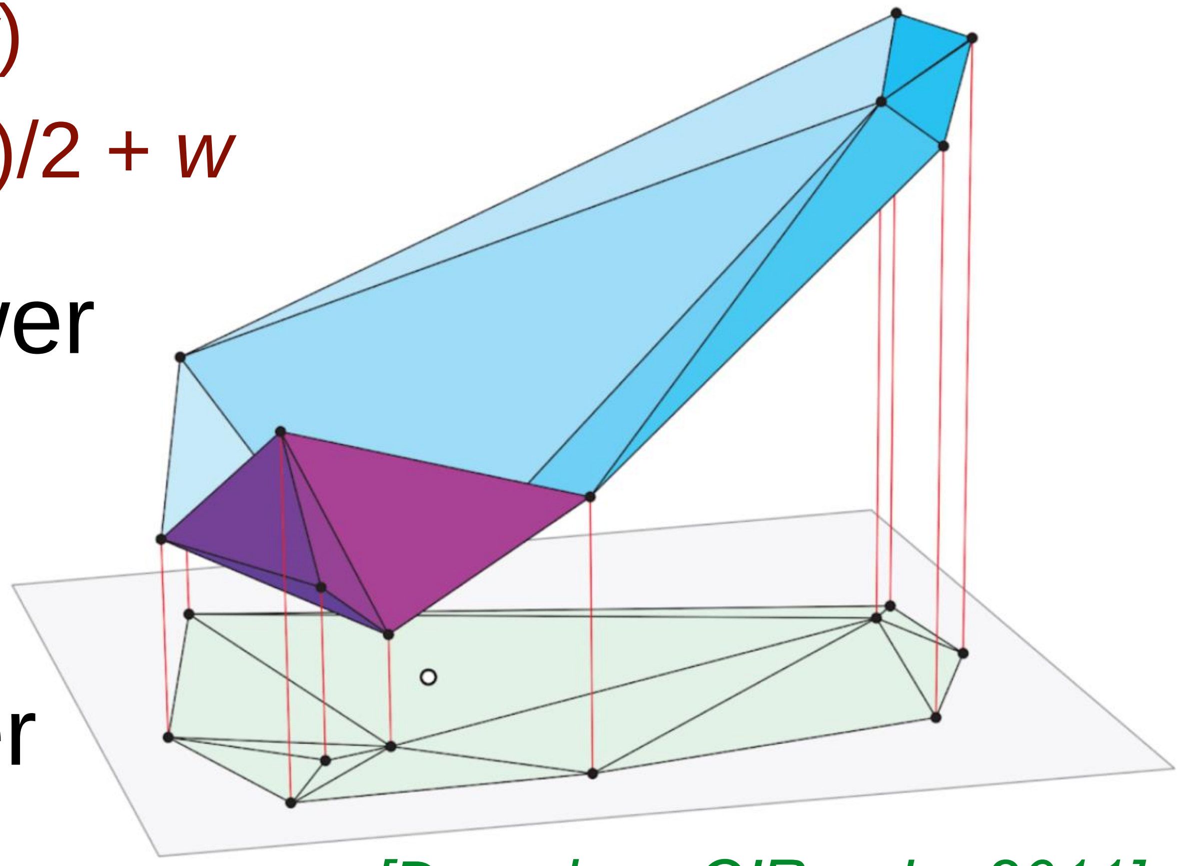




(Weighted) Delaunay/Voronoi lifting

- For any weighted point $p = ((a,b),w)$ in the plane, define
  - Lifted point $p' = (a, b, (a^2+b^2)/2 - w)$
  - Dual plane p^*: $z = ax + by - (a^2+b^2)/2 + w$
- Delaunay(P) = projection of lower convex hull of P'
  - "regular / coherent subdivision"
- Voronoi(P) = projection of upper envelope of P^*
  - "power / Laguerre diagram"

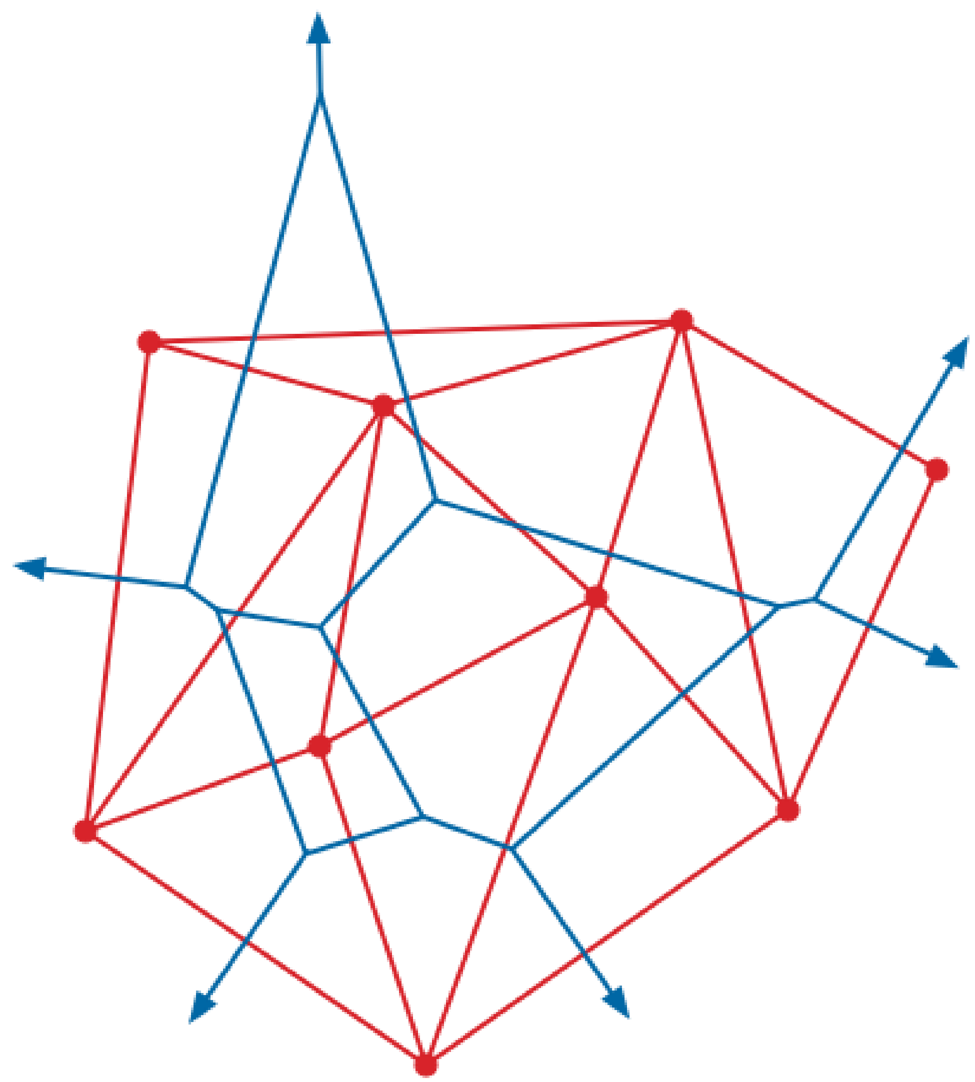


[Devadoss O'Rourke 2011]

[Brown 1980, Seidel 1982, Edelsbrunner Seidel 1985]

Reciprocal = Voronoi

- Voronoi and Delaunay are orthogonal ➔ reciprocal
- Every reciprocal diagram is a weighted Voronoi diagram
  - Lifting = Lifting



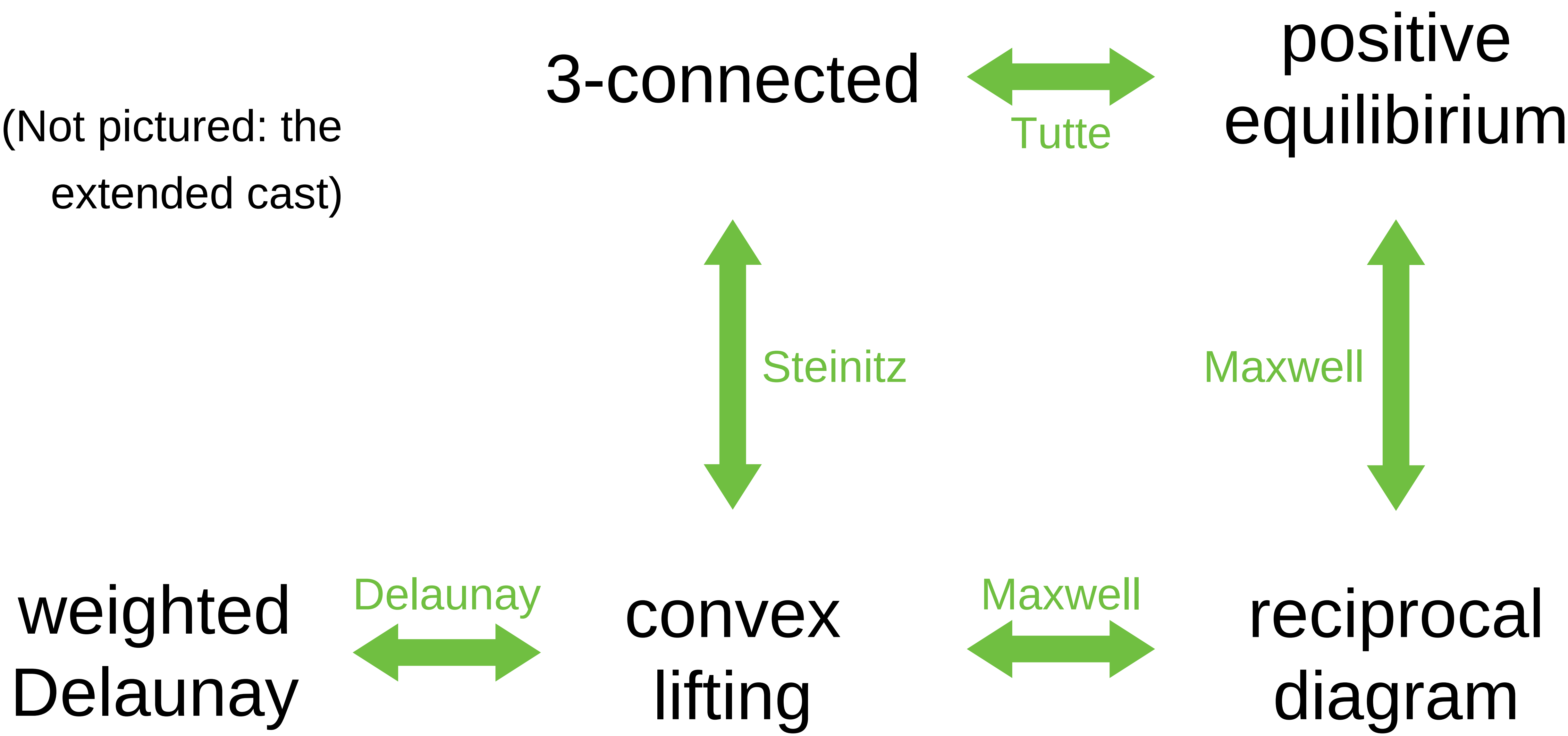
Maxwell-Cremona-Delaunay Correspondence

For any plane graph *G* with a convex outer face, the following are (essentially) equivalent:

- Positive equilibrium stress ω
- Reciprocal embedding of *G**
- Convex lifting
- Delaunay vertex weights

Tutte + Maxwell + Delaunay

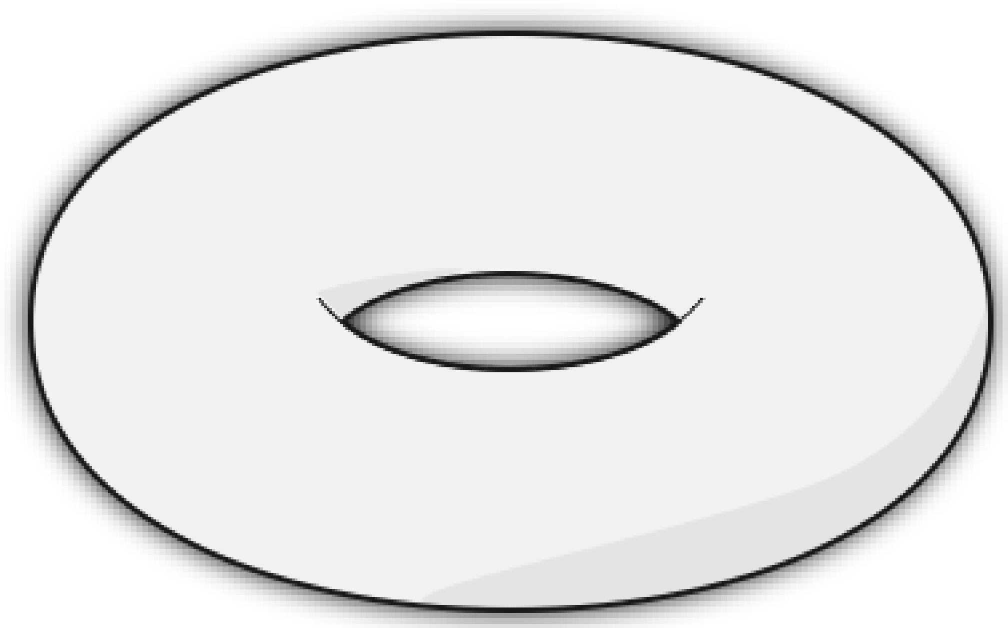
(Not pictured: the
extended cast)



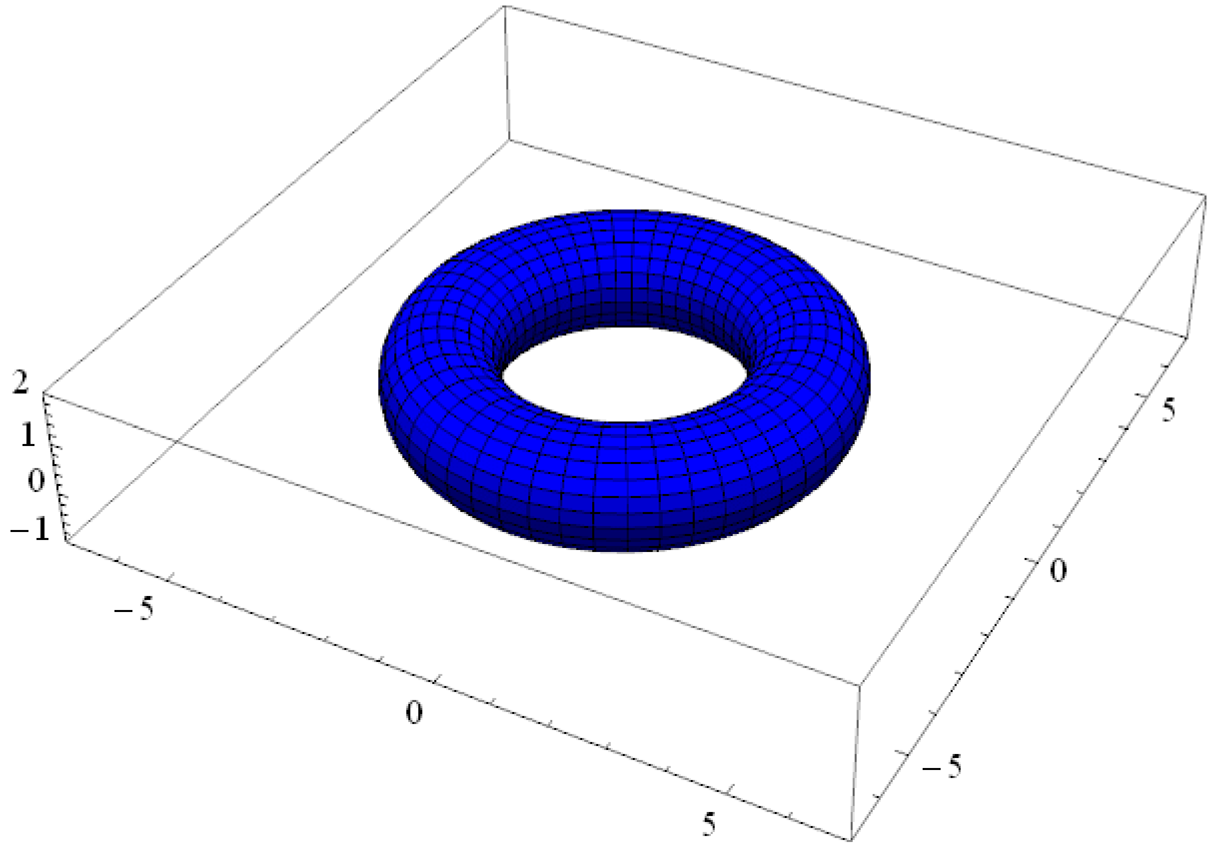
What happens on surfaces?

> Finally we may remark that very little is known about representations of graphs in the projective plane and higher surfaces (4).

- Today: flat tori

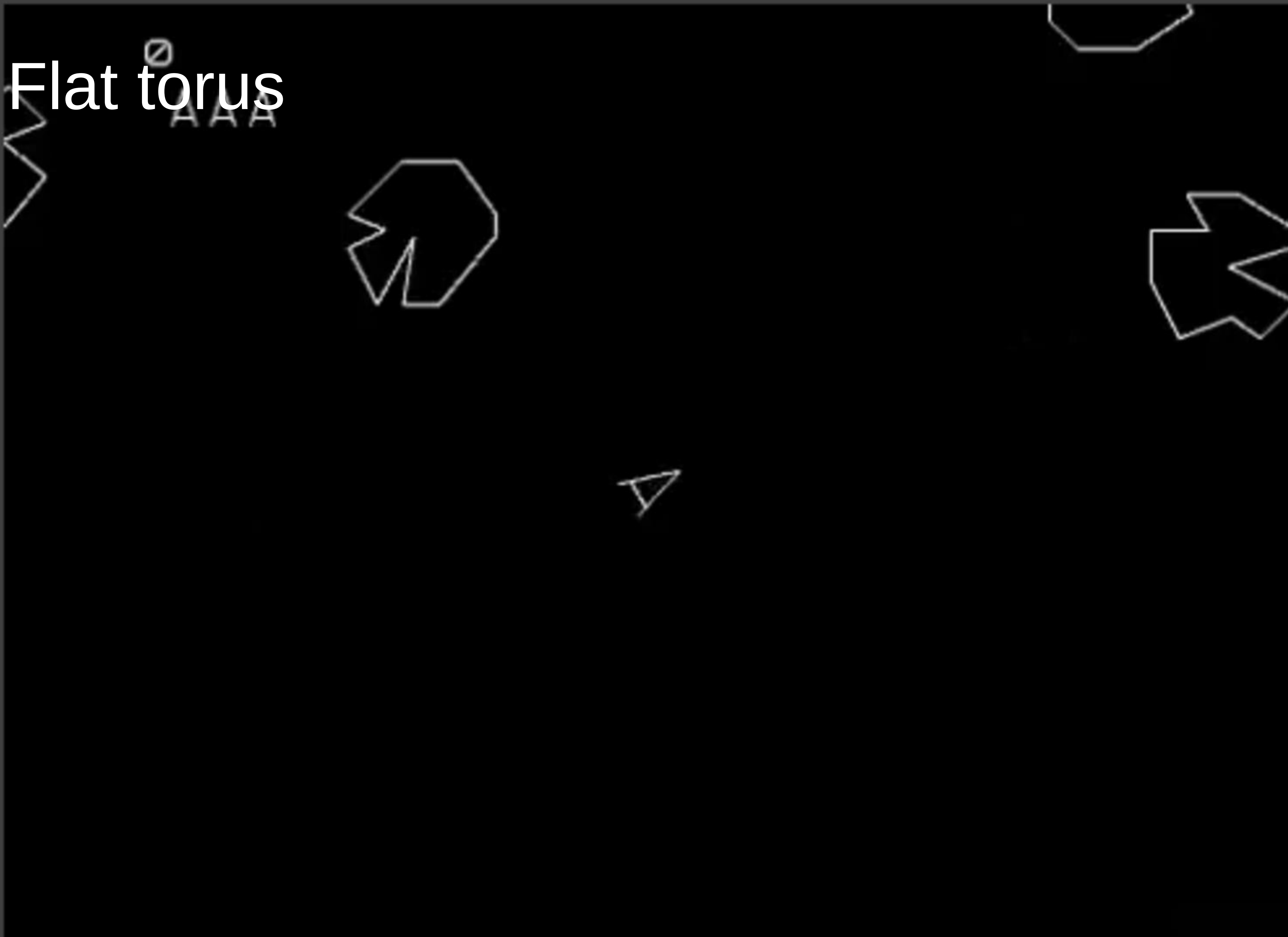


Flat torus



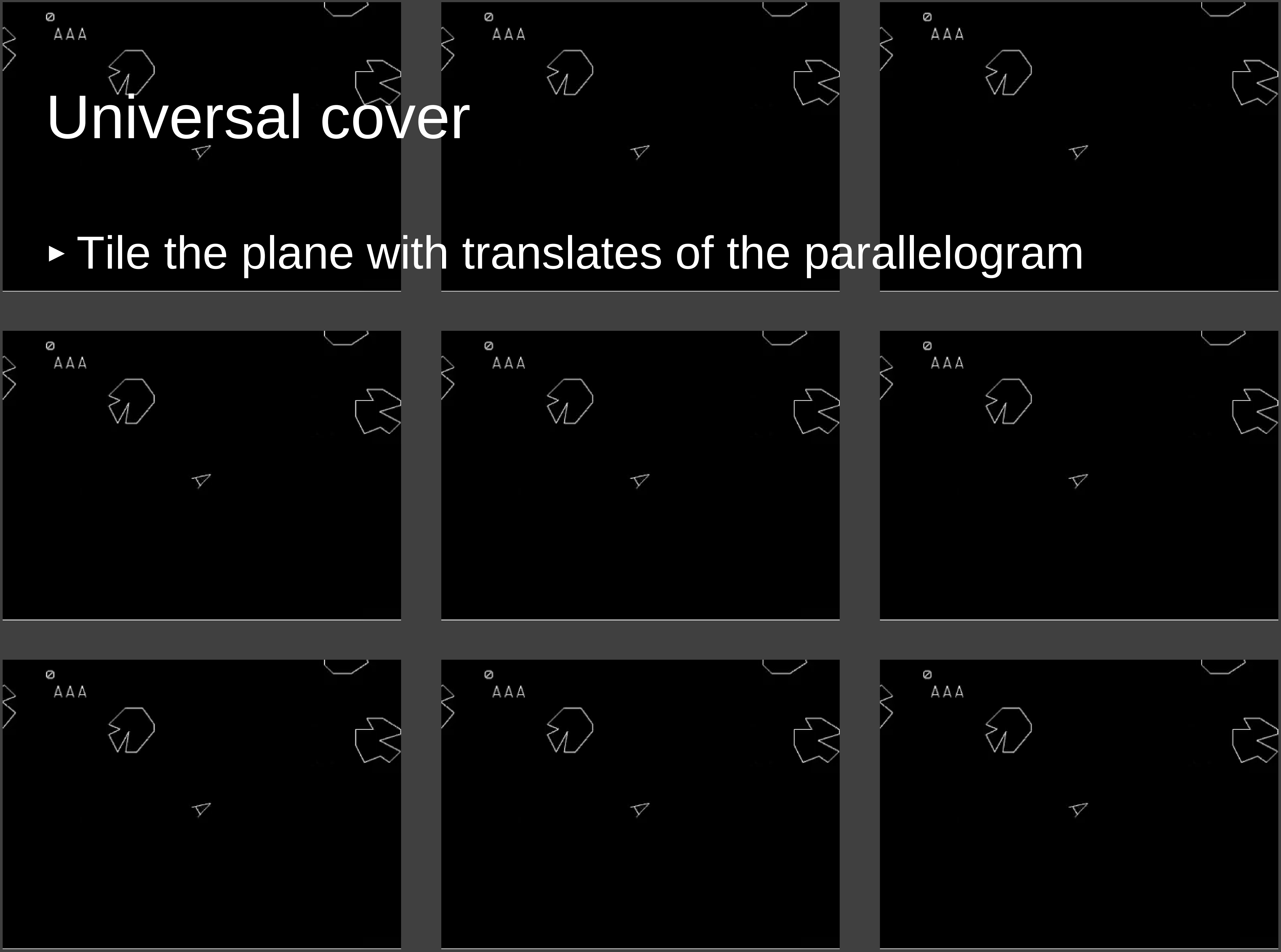
[Podkalicki (Mathematica.SE)]

Flat torus



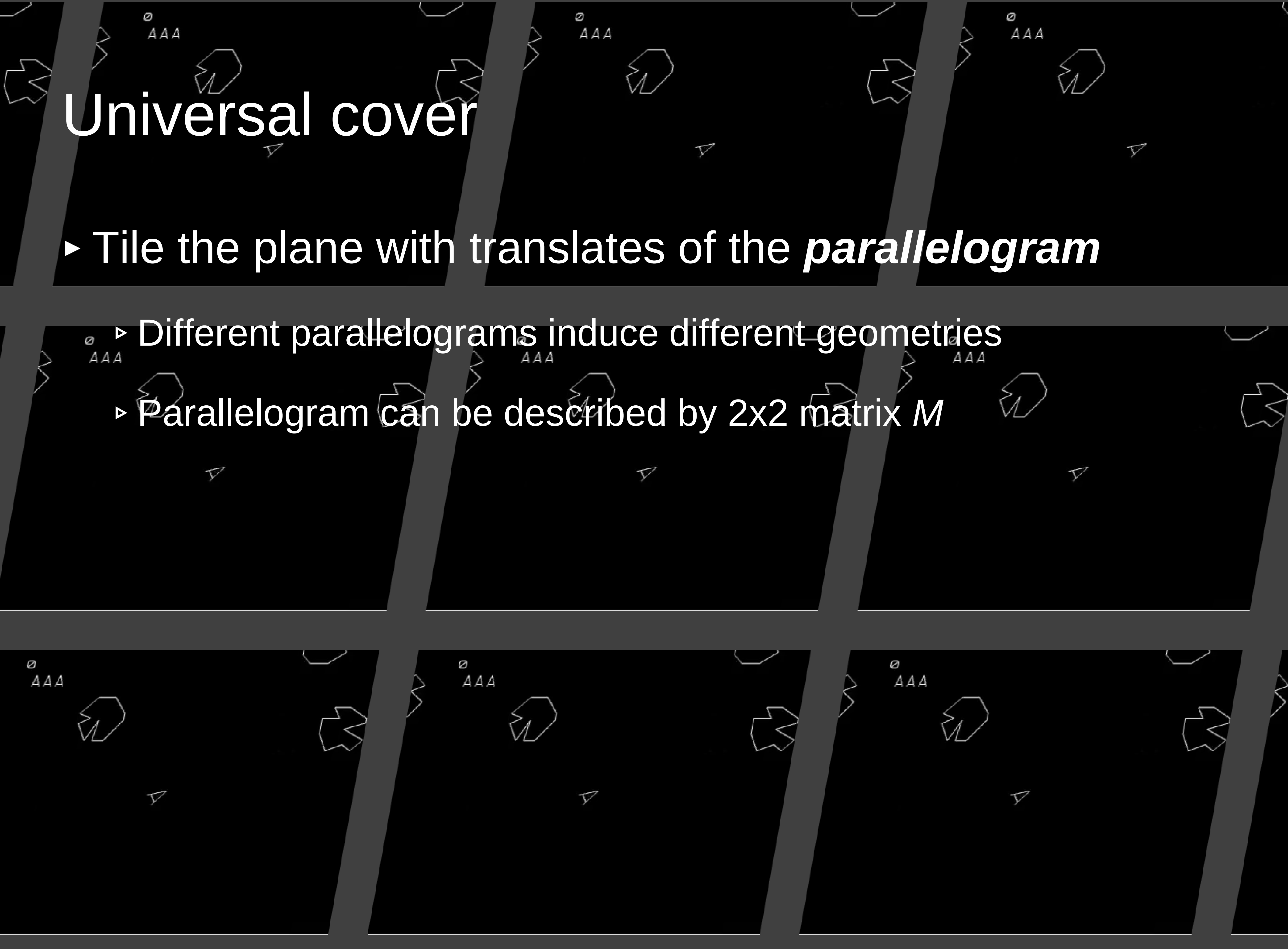
Universal cover

- Tile the plane with translates of the parallelogram



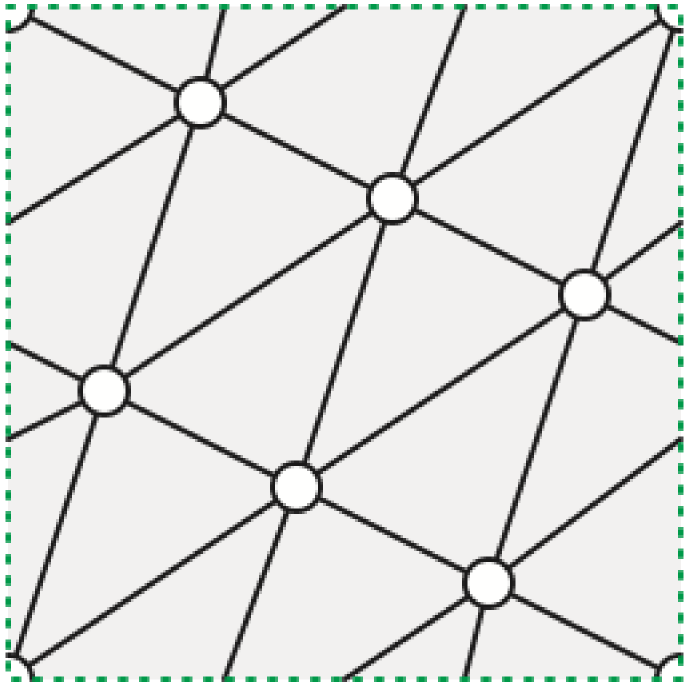
Universal cover

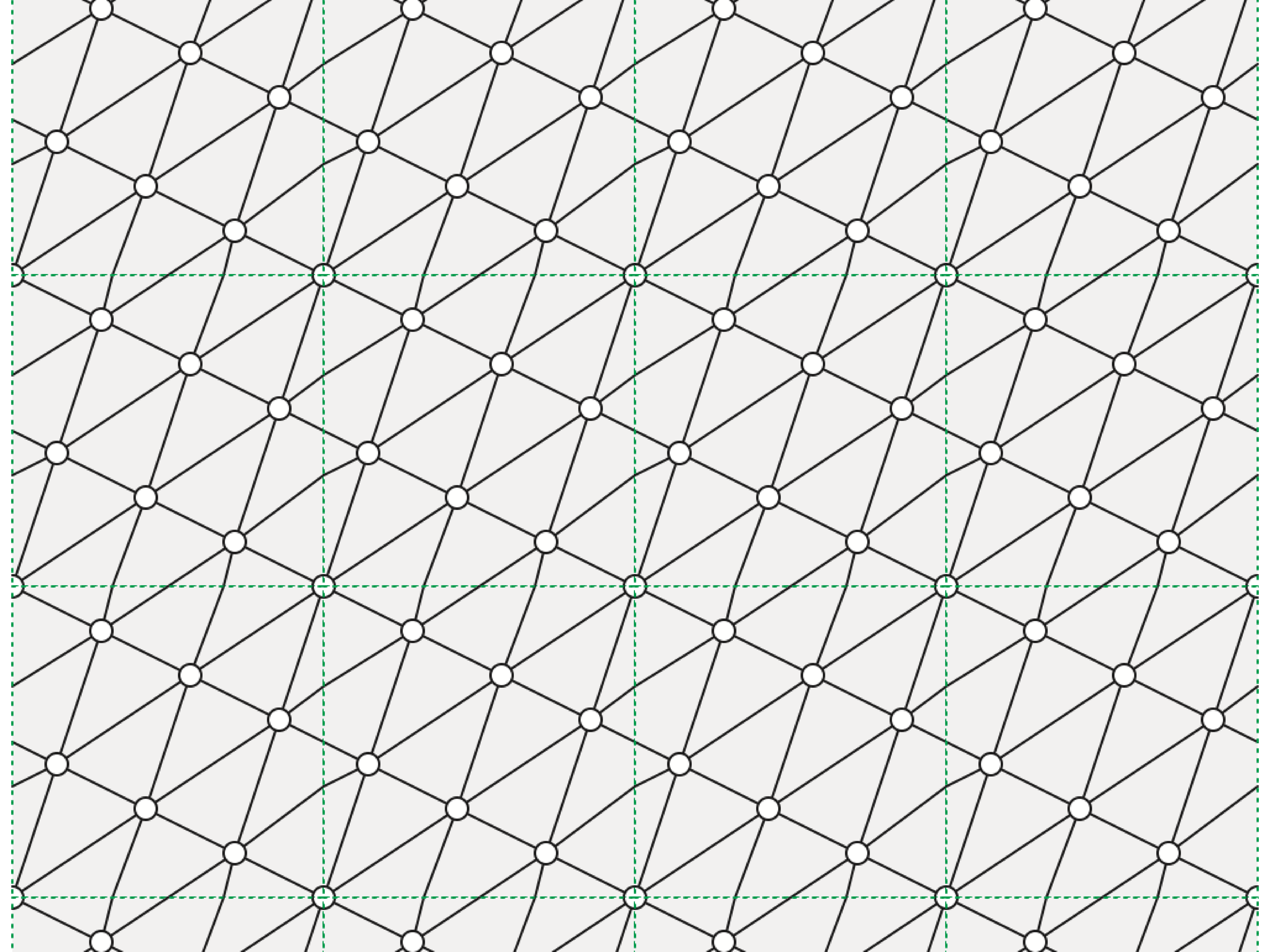
- Tile the plane with translates of the ***parallelogram***
  - Different parallelograms induce different geometries
  - Parallelogram can be described by 2x2 matrix *M*



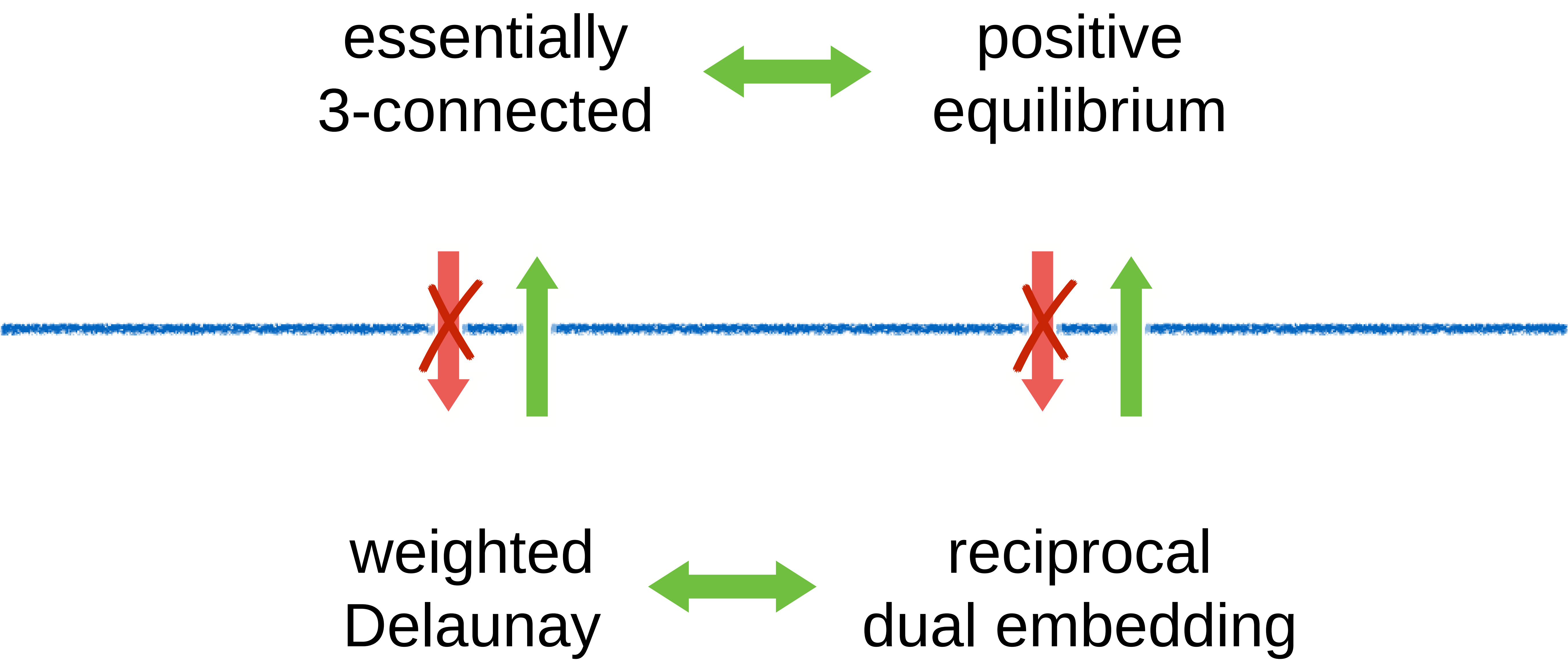
Flat torus graphs

- Geodesics are "straight lines"
- Torus graph = geodesic embedding on some flat torus
  - = lifts to infinite plane graph in universal cover





On the flat torus

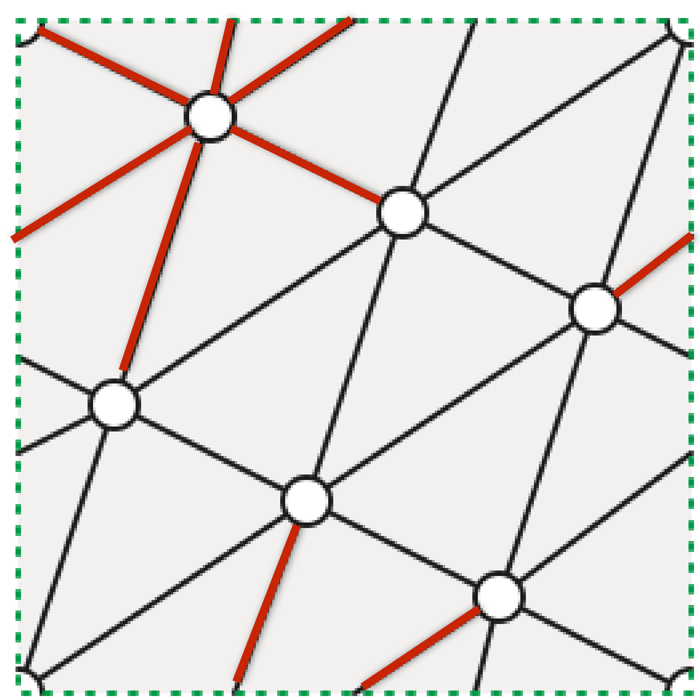


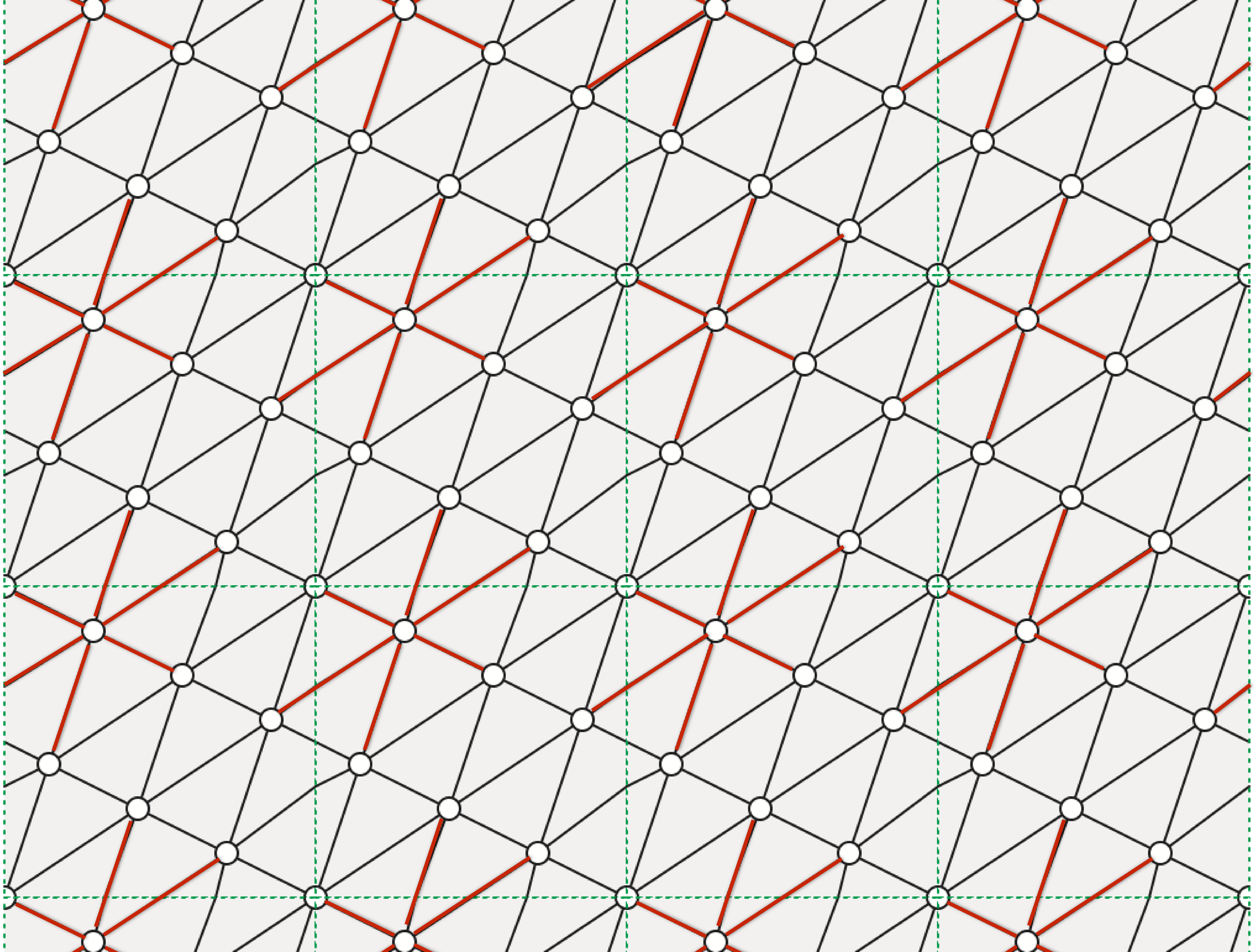
Equilibrium is local

- ω is an equilibrium stress for G if and only weighted edge vectors around each vertex sum to zero.

$$\sum_v \omega_{uv}(\hat{v} - \hat{u}) = \sum_v \omega_{uv}(v - u + [u \rightarrow v]) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Equivalently, universal cover $(\hat{G}, \omega)$ is in equilibrium

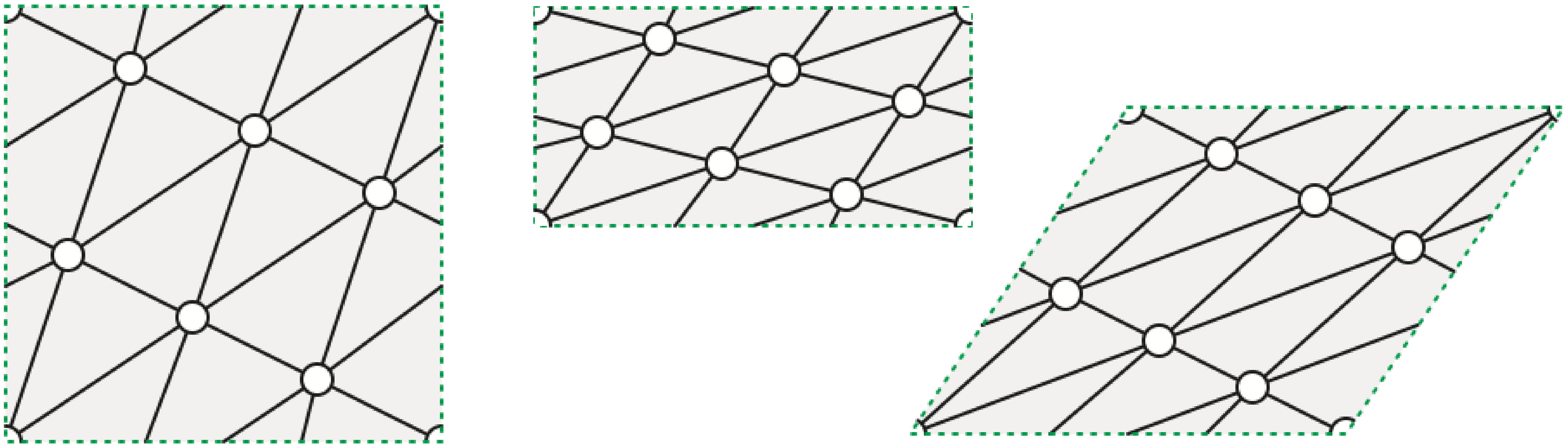




Equilibrium is shape-agnostic

- If ω is an equilibrium stress for *G* on *any* flat torus, then ω is an equilibrium stress for *G* on *every* flat torus.

$$\sum_v \omega_{uv} \cdot (M\hat{v} - M\hat{u}) = M \cdot \sum_v \omega_{uv} \cdot (\hat{v} - \hat{u}) = M\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$



What about Tutte?

Every *essentially* 3-connected graph on any flat torus is *isotopic to* a positive equilibrium embedding

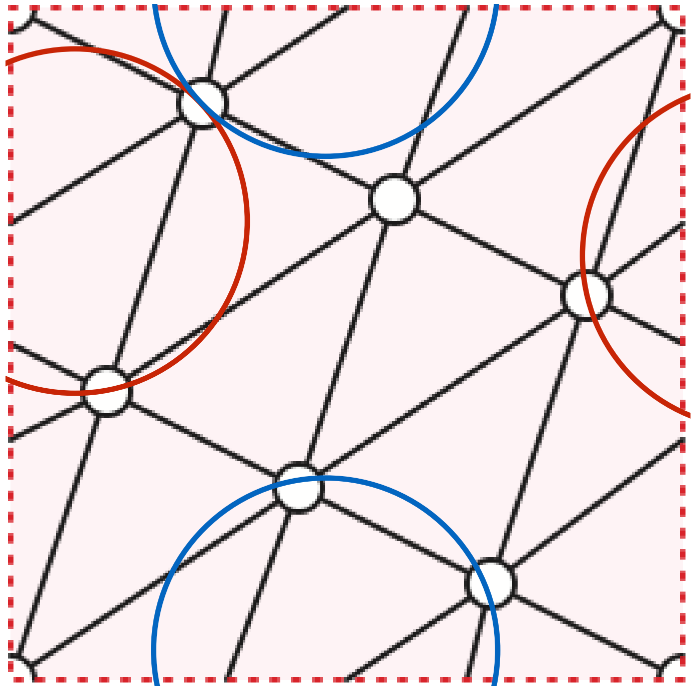
- Unique (*up to translation*) for *any* positive stress vector ω>0.
- Isotopic drawing of *G* minimizing $\Phi(G,\omega) := \sum_e \omega_e \cdot |e|^2$
- Solution to Laplacian linear system

$$\sum_v \omega_{uv}(\hat{v}-\hat{u}) = \sum_v \omega_{uv}(v-u+[u\to v]) = \binom{0}{0}$$

[Y. Colin de Verdière 1990,
Lovász 2004, Steiner Fischer 2004,
Gortler Gotsman Thurston 2006]

Delaunay is local

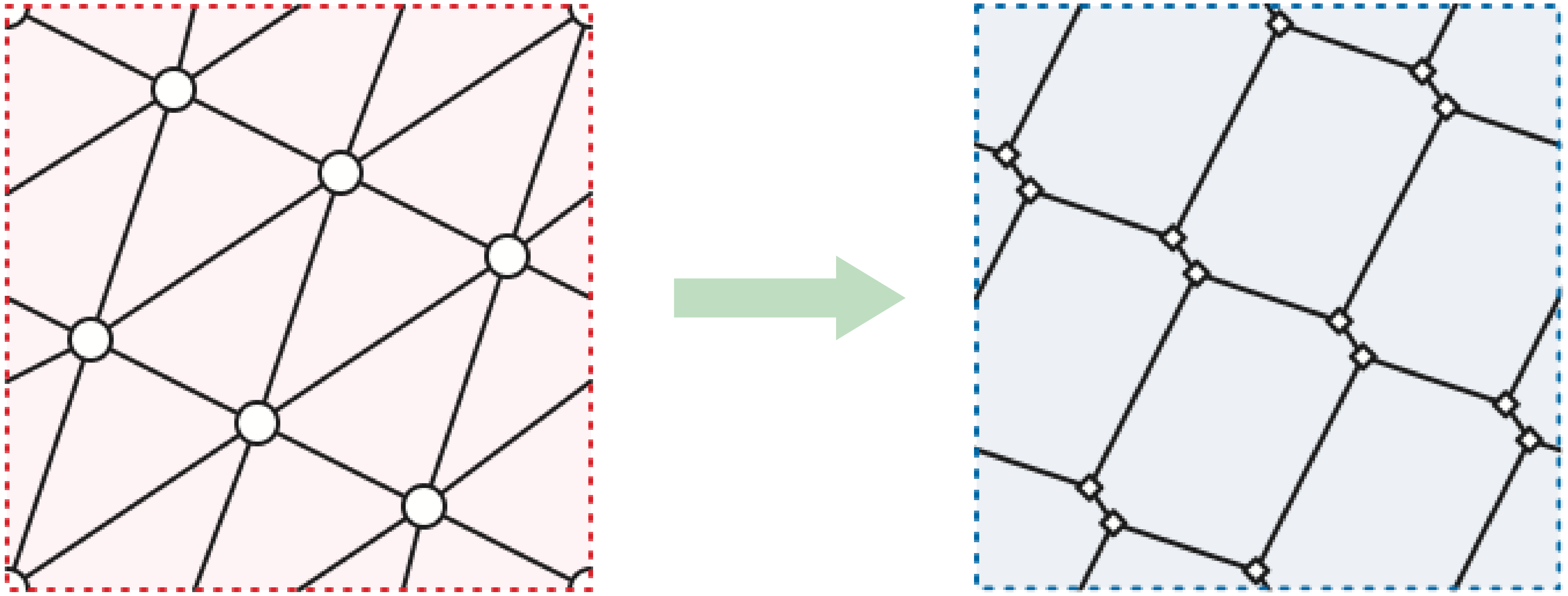
- For *fixed* vertex weights, G is Delaunay iff every edge is locally Delaunay *[Bobenko Springborn 2005]*



Reciprocal diagrams

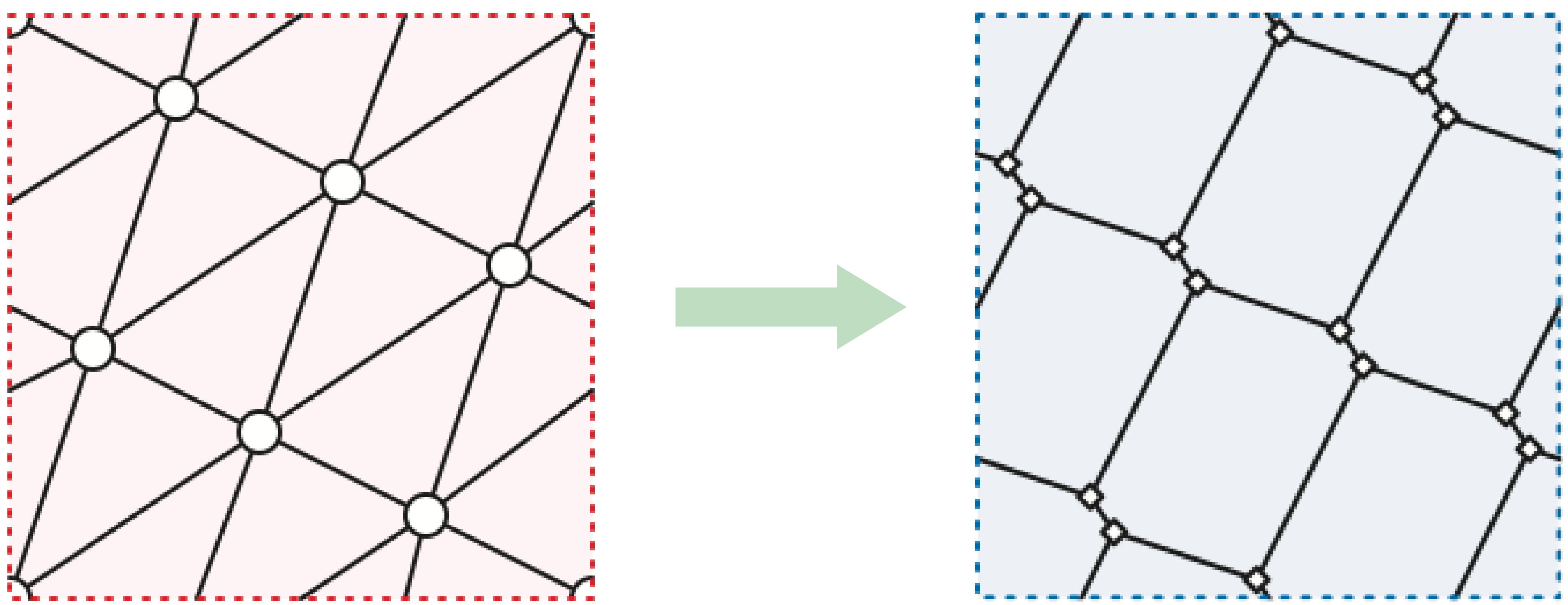
Geodesic embedding of *G** on *the same* flat torus as *G*

$$e \perp e^*$$



Delaunay ⇔ reciprocal

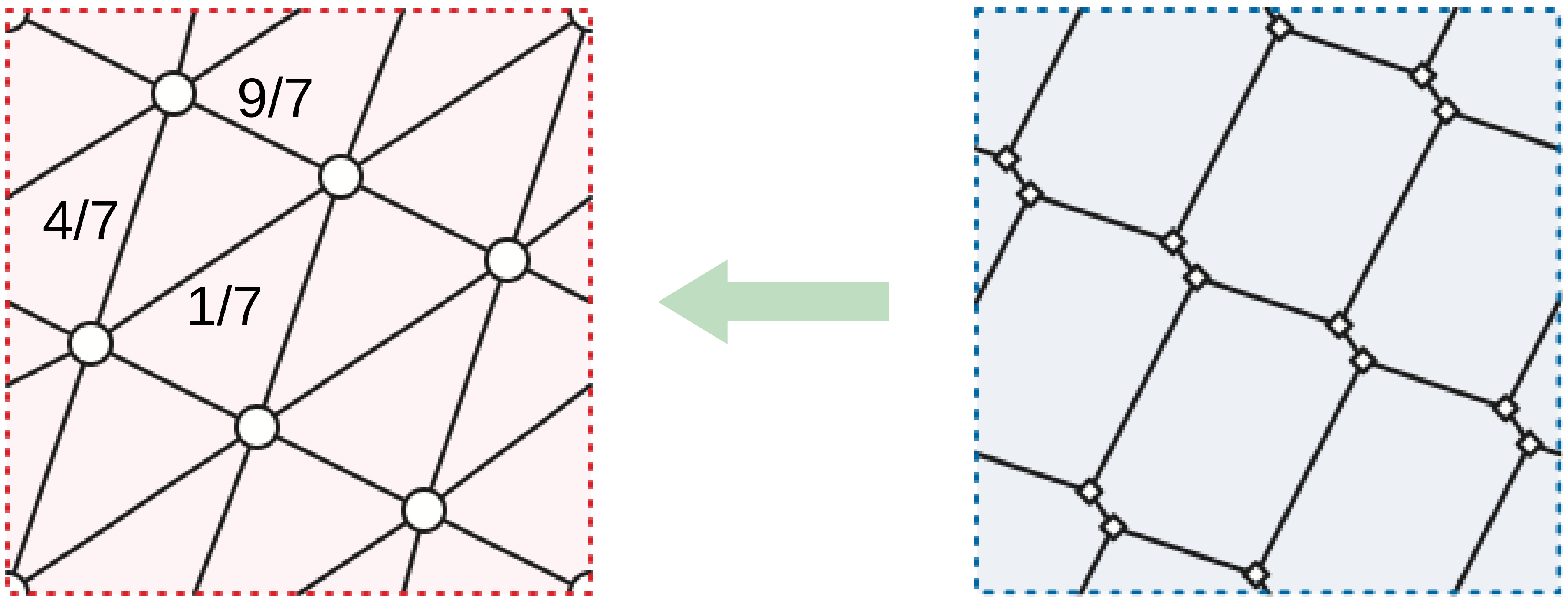
- Any vertex weights that make *G* Delaunay define a reciprocal diagram *G** and vice versa.



Delaunay *Voronoi*

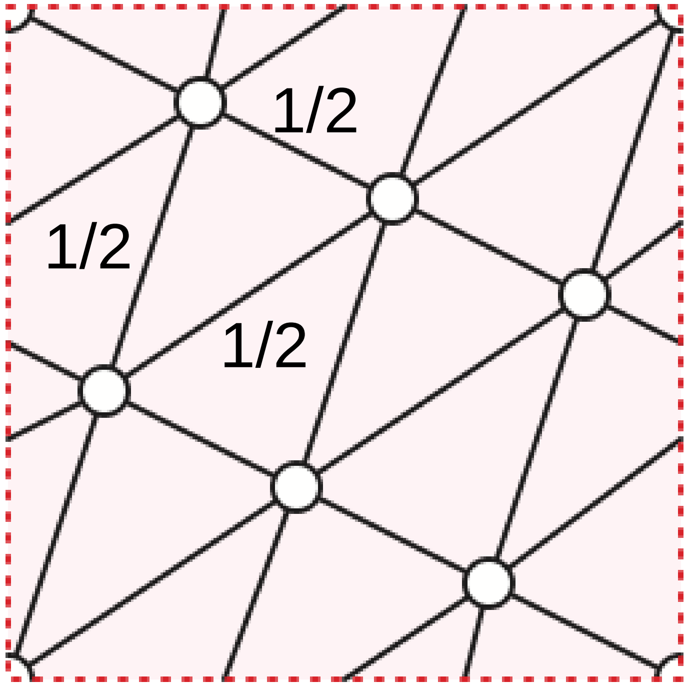
Reciprocal ⇒ equilibrium

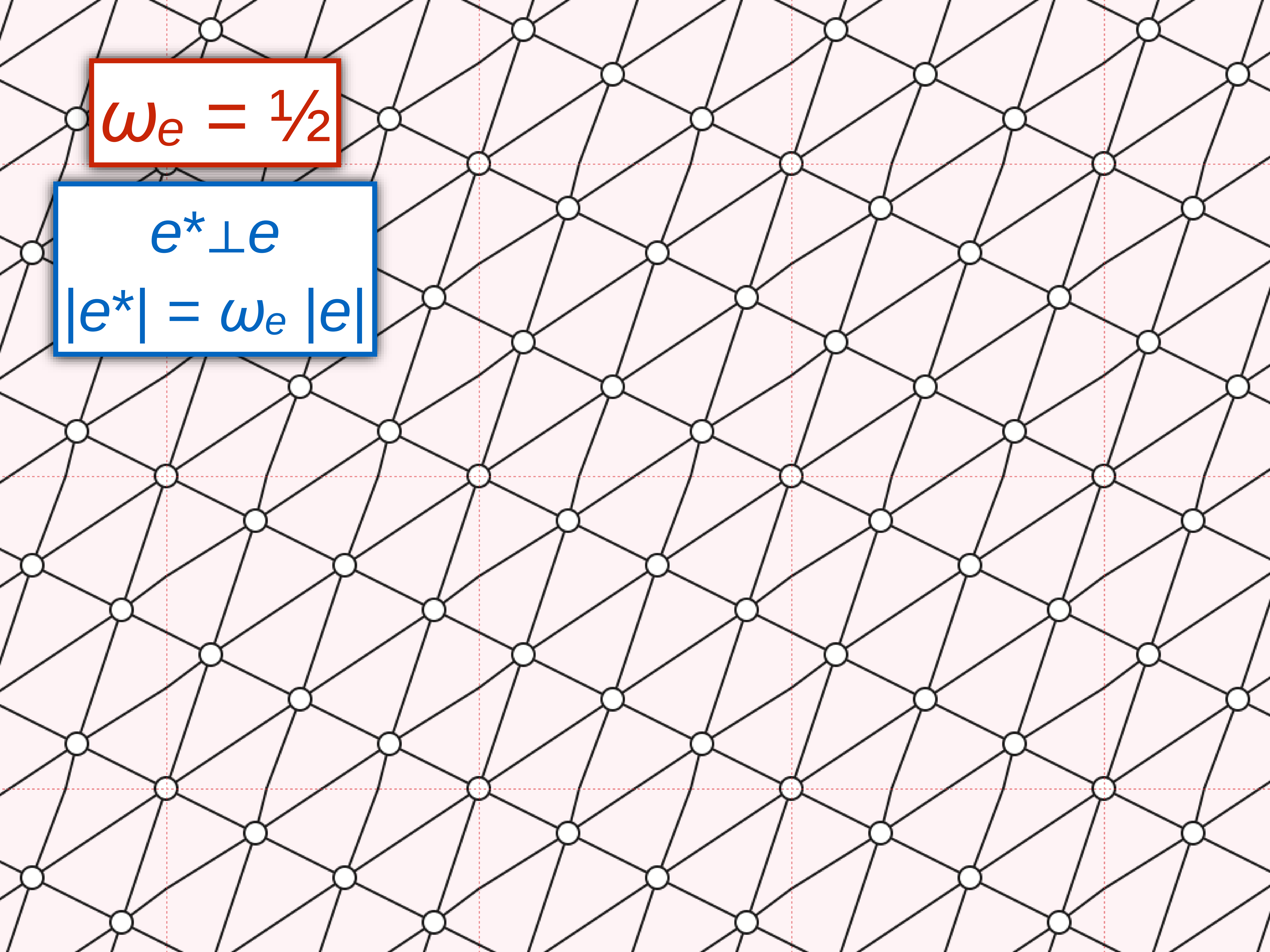
- Any reciprocal diagram defines an equilibrium stress ω where $\omega_e = |e^*| / |e|$.

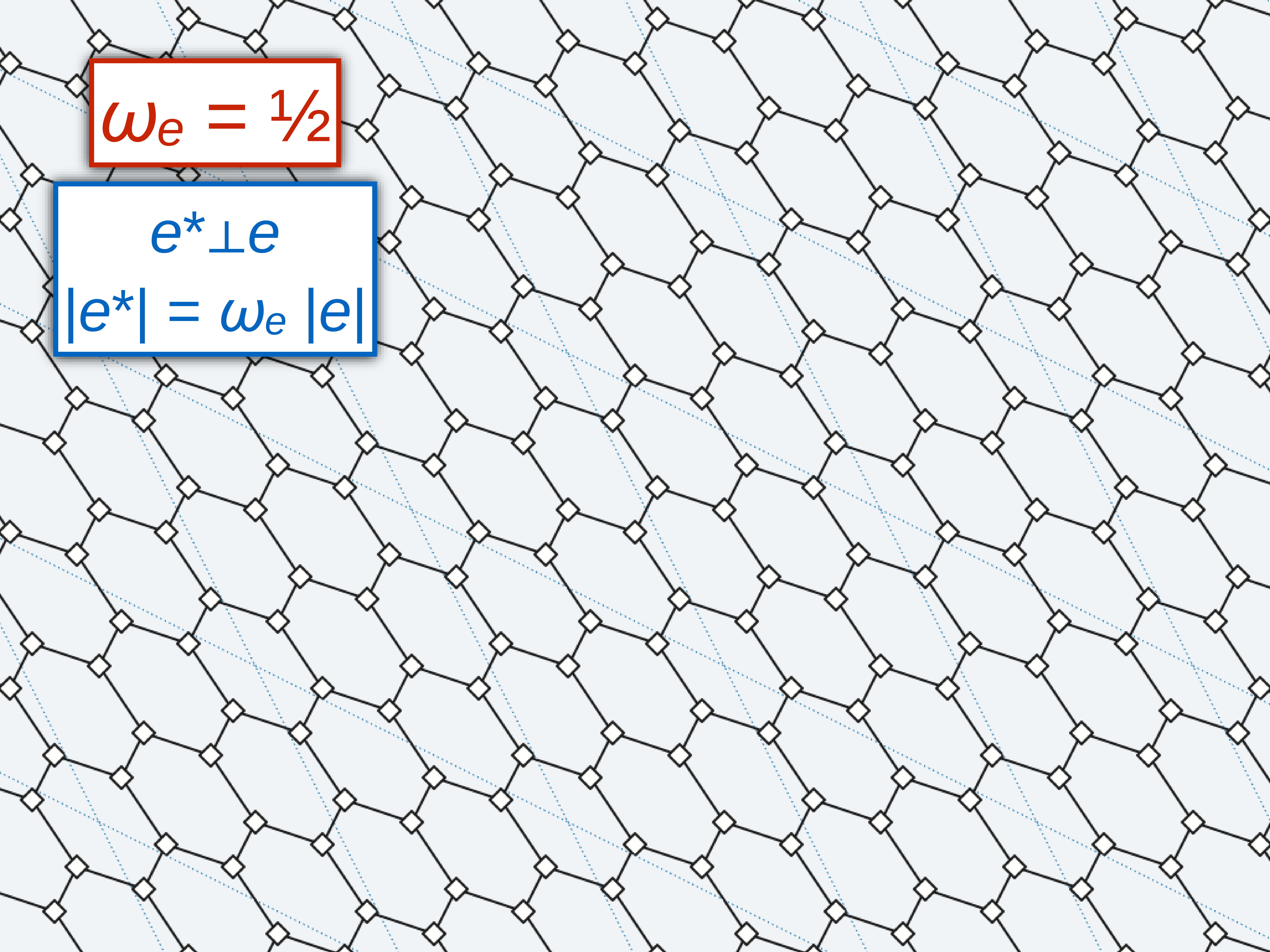


Equilibrium ⇏ reciprocal

[Erickson, L.]



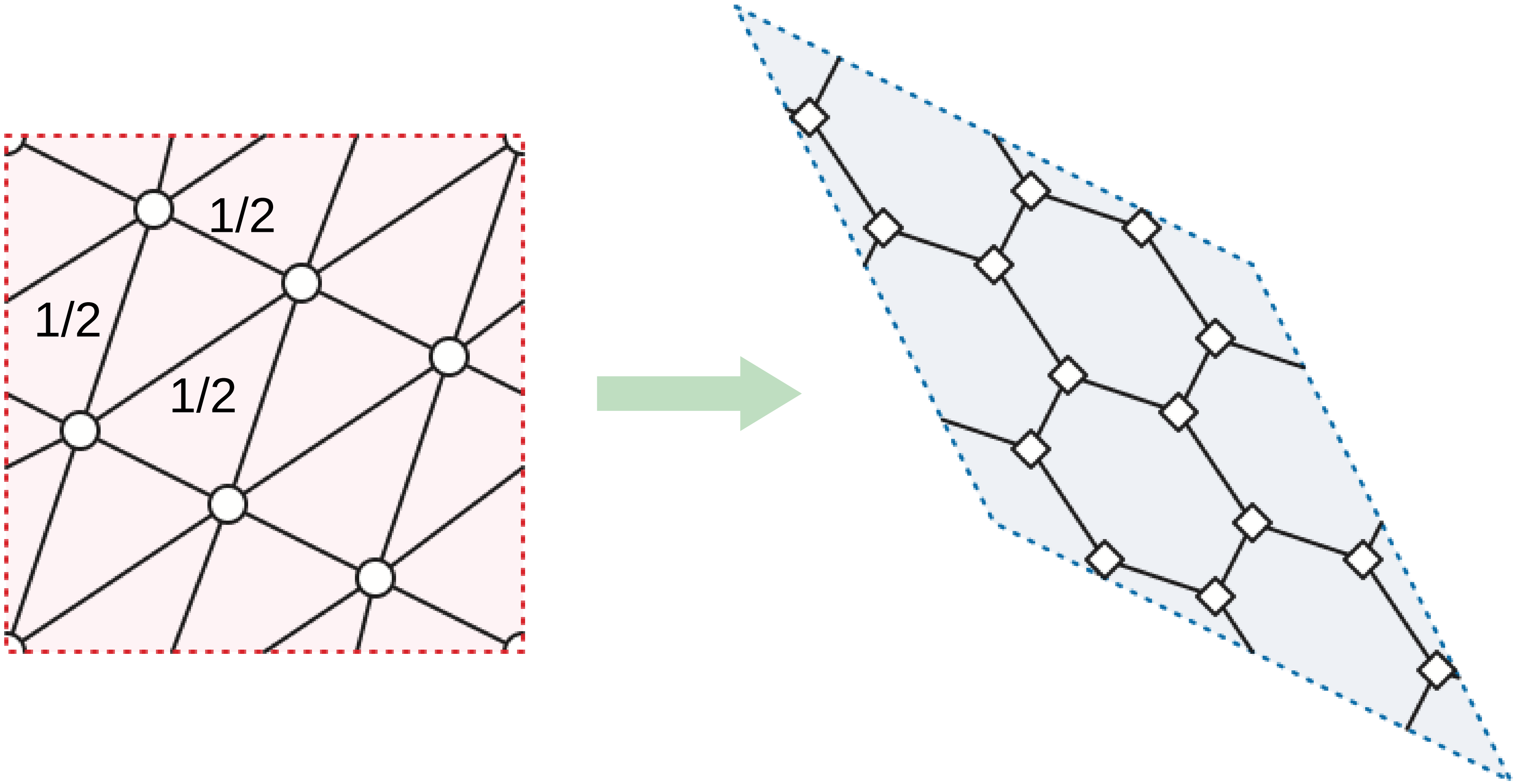

$\omega_e = ½$
$e^* \perp e$
$|e^*| = \omega_e \, |e|$


$\omega_e = ½$
$e^* \perp e$
$|e^*| = \omega_e\,|e|$

Equilibrium ⇏ reciprocal

[Erickson, L.]

- An equilibrium stress ω does *not* necessarily define a reciprocal diagram with $\omega_e = |e^*| / |e|$
  - Reciprocality is ***NOT*** shape agnostic!



Isotropy parameters

[Erickson, L.]

- Fix a graph *G* on the *unit square* flat torus
- Any equilibrium stress ω for *G* defines three parameters:

$$\alpha := \sum_e \omega_e \Delta x_e^2$$

$$\beta := \sum_e \omega_e \Delta y_e^2$$

$$\gamma := \sum_e \omega_e \Delta x_e \Delta y_e$$

- ω is a reciprocal stress for *G* if and only if $(\alpha,\beta,\gamma) = (1,1,0)$

Isotropy conditions

[Erickson, L.]

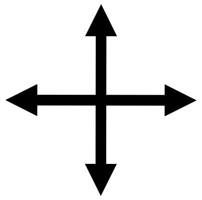
- Equivalently, ω is a reciprocal stress for G on the unit square flat torus if and only if

$$\sum_e \omega_e \cdot (\Delta x_e^2 + \Delta y_e^2) = 2$$

Tutte energy (scale)

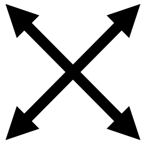
$$\sum_e \omega_e \cdot (\Delta x_e^2 - \Delta y_e^2) = 0$$

Orthogonal anisotropy



$$\sum_e \omega_e \cdot \Delta x_e \cdot \Delta y_e = 0$$

Diagonal anisotropy



Isotropy condition (proof)

[Erickson, L.]

- For any cocycle in a reciprocal diagram *G**, the sum of its displacement vectors must equal its *homology class*.
- Suffices to check two cycles in a *homology basis* of *G**
- Four constraints (*x* and *y* for two cycles), but one is redundant
- Displacement vectors in G define homology basis of *circulations* in *G**

$$\alpha := \sum_e \omega_e \Delta x_e^2$$

$$\beta := \sum_e \omega_e \Delta y_e^2$$

$$\gamma := \sum_e \omega_e \Delta x_e \Delta y_e$$

Isotropy condition (proof)

[Erickson, L.]

- *Blah blah blah homology class.*
- *Blah blah blah two cycles blah blah blah homology basis.*
- *Blah blah four constraints blah blah one redundant.*
- *Blah blah displacement blah blah circulations blah blah.*

$$\alpha := \sum_e \omega_e \Delta x_e^2$$

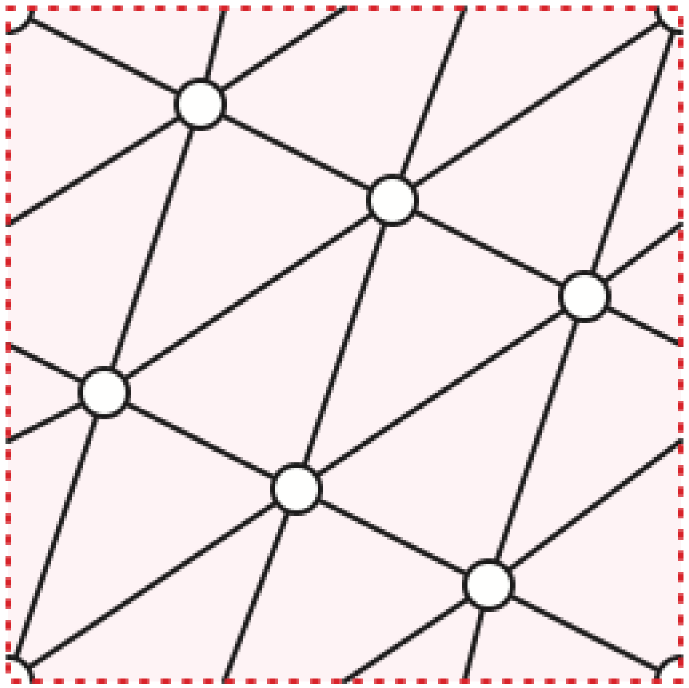
$$\beta := \sum_e \omega_e \Delta y_e^2$$

$$\gamma := \sum_e \omega_e \Delta x_e \Delta y_e$$

Reciprocal conditions

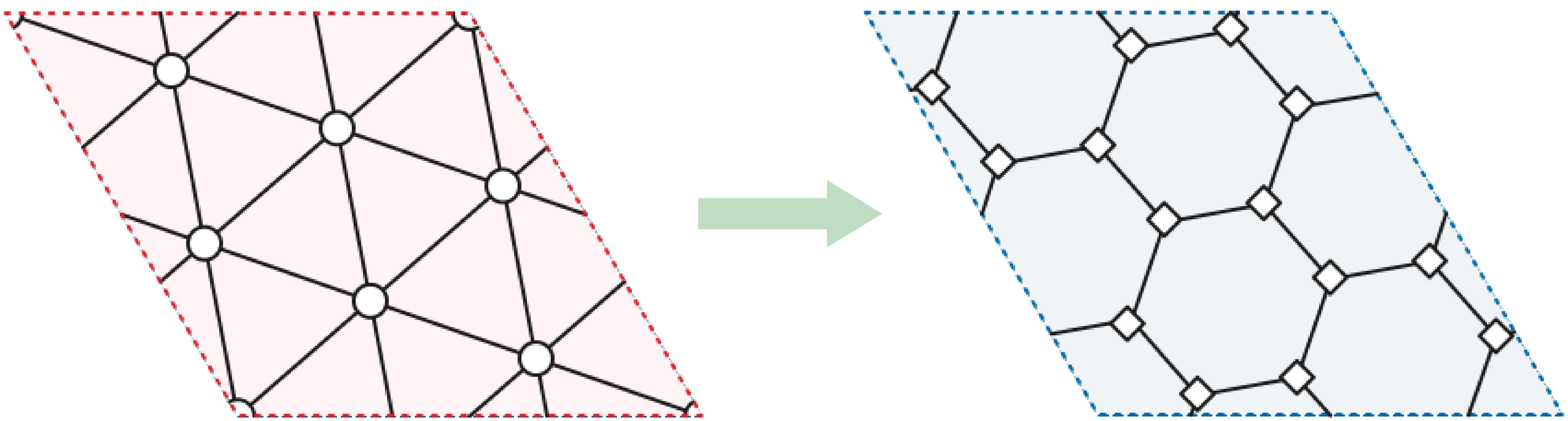
[Erickson, L.]

- ω is reciprocal for G on *some* flat torus iff $\alpha\beta - \gamma^2 = 1$.
  - This is just a scaling condition.
- If $\alpha\beta - \gamma^2 = 1$, then ω is reciprocal for G on any flat torus similar to T_M, where $M = \begin{bmatrix} \beta & -\gamma \\ 0 & 1 \end{bmatrix}$.



$$\omega_e = 1 \quad\Downarrow\quad \alpha = \beta = 2 \qquad \gamma = 1 \quad\Downarrow\quad \alpha\beta - \gamma^2 = 3$$

$$\omega_e = 1/\sqrt{3} \quad\Downarrow\quad \alpha = \beta = 2/\sqrt{3} \qquad \gamma = 1/\sqrt{3} \quad\Downarrow\quad \begin{bmatrix} 2/\sqrt{3} & -1/\sqrt{3} \\ 0 & 1 \end{bmatrix}$$



Conclusion

- Every essentially 3-connected torus graph is homotopic to a weighted Delaunay complex on *some* flat torus.
    - This also follows from generalizations of Koebe-Andreev-Thurston circle packing. *[Y. Colin de Verdière 1991, Mohar 1997]*
    - But we get *all possible* Delaunay embeddings.

Open questions

- What happens on more complicated surfaces?
  - Spring embeddings work (at least for simplicial complexes)
    [Y. Colin de Verdière 1990]
  - Delaunay triangulations work (at least for simplicial complexes)
    [Bogdanov Deviller Ebbens Iordanov Teillaud Vegter... 2014–2020]
  - But how are the two related?
- Is this good for anything?