The pure condition

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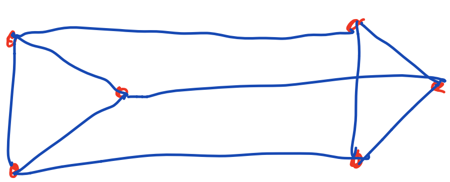
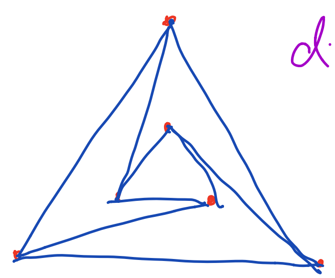
ref: "The algebraic geometry of stresses in frameworks"
White & Whiteley (SIAM Alg. Disc. Methods '83)

Defn: Framework $(G, \underline{p})$ in dim d

- $G = (V, E)$ graph $|V| = n$, $|E| = m$ finite, simple, undirected
- $\underline{p} = (p_1, \ldots, p_n) \in (k^d)^n$ [$k = \mathbb{R}$ or $\mathbb{C}$]
- d dimension

$d = 2$





Defn: A framework $(G, \underline{p})$ in dim d is *infinitesimally rigid* if the system

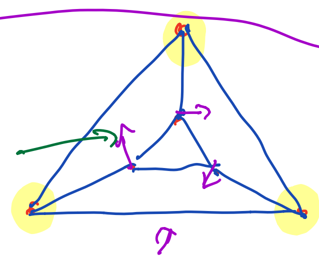
$$\langle \overset{\text{fixed}}{p_j - p_i}, \overset{\text{variables}}{\dot{p}_j - \dot{p}_i} \rangle = 0$$

[Rigidity matrix $R(\underline{p})$ is the $dn \times m$ matrix of the system]

has rank $dn - \binom{d+1}{2}$. Solutions $\dot{p}$ called *infinitesimal flexes*. Always a $\binom{d+1}{2}$-dim'l space of *trivial flexes*

either all generic $(G, \underline{p})$ inf rigid or all are inf flex.

$\dot{p}_i$

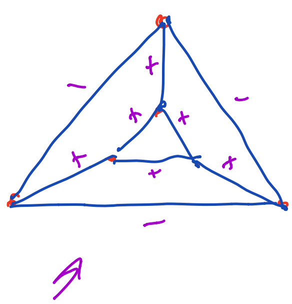


- implies rigidity
- generically equ. to it.
- is a *generic property*
- G is "GLR" if $(G, \underline{p})$ generic rigid.

Defn: An *equilibrium stress* $\omega \in \mathbb{R}^m$ is an assignment of weights to edges of $(G, \underline{p})$ so that

$$\forall i \in V \quad \sum_{j \sim i} \omega_{ij} (p_j - p_i) = 0$$

← vertex is in equilibrium



- Dual to infinitesimal rigidity
- "Index theorem"

$$dn - m = \dim(\text{flexes}) - \dim(\text{stresses})$$

- If $m = dn - \binom{d+1}{2}$ ← minimal number of edges for inf. rigidity

$(G, \underline{p})$ inf rigid

[Maxwell] $\Longleftrightarrow$

$(G, \underline{p})$ has no eq. stress

Question: If G is GLR and $m = dn - \binom{d+1}{2}$ [$d=2$: $2n-3$] for which p is (G,p) infinitesimally flexible?

[equ. (G,p) has an eq. stress?] "special positions"

These are interesting because they can have other nice properties.

Approach of White & Whiteley:

Can be addressed using invariant theory

Basic observation: If ω is an eq. stress of $(G,\underline{p})$, also for $(G, \alpha(\underline{p}))$ $\forall$ projective maps α.

Affine: $\sum_{j\sim i} \omega_{ij}(Ap_j + \cancel{b} - Ap_i - \cancel{b}) = A\left(\sum_{j\sim i} \omega_{ij}(p_j - p_i)\right)$

lhs $=0$ $\iff$ rhs $=0$

Thm: ("First thm of invariant theory") If

$P(x_1, \ldots, x_n)$ is a polynomial in $x_n \in \mathbb{R}^d$ ($\mathbb{K}^d$) the coords of vectors $x_1, \ldots, x_n \in \mathbb{K}^d$ s.t.

$$P(\underline{X}) = 0 \iff P(\alpha(\underline{X})) = 0 \quad \forall \alpha \in SL(k,d)$$

Then P has a representation as a polynomial in "brackets"

$$[\underbrace{x_{i_1} \cdots x_{i_d}}_{d \text{ of the vectors in } \underline{X}}] = \det\begin{pmatrix} 1 & & 1 \\ x_{i_1} & \cdots & x_{i_d} \\ 1 & & 1 \end{pmatrix}$$

- In the rigidity application

$$\boxed{e_{ij} = p_j - p_i} \text{ "edge vectors"}$$

convenient choice.

Thm (White-Whiteley '83): If G is GLR and $m = dn - \binom{d+1}{2}$ the set of "special positions" configurations p s.t. (G, p) inf. flexible is cut out by a single bracket polynomial. (in $d+1$ vectors $\hat{p}_i = (p_i; 1)$ if you want point " d edge vectors)

- Called the "pure condition" C_G.

Sketch: Use the 1st thm: special positions are projectively invariant. (You can gain but not lose stresses.)

- Some collection of bracket polys work. ← (hard part: want just 1)
- Main point: all flexes/triv.

$$R(p) : k^{dn}/\text{triv} \xrightarrow{\text{linear}} k^{dn - \binom{d+1}{2}}$$

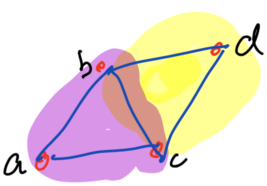
$$\cong \qquad k^{dn - \binom{d+1}{2}} \longrightarrow k^{dn - \binom{d+1}{2}}$$

rigidity matrix

[C_G = det of this map]

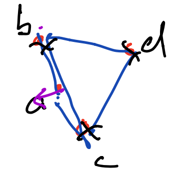
Application 1: Factors of C_G

Ex:

$G =$  $d = 2$

$C_G = [a\,b\,c][b\,c\,d]$

inf flex ⟺ has a flat △.



Thm (W+W): If $G' \subseteq G$ is a GLR subgraph, then $C_{G'}$ is a factor of C_G.

Sketch: If $(G', p|_{V(G')})$ supports an eq. stress in (G, p) then (G, p) does too. Hence $C_G = \langle C_{G'} \rangle$.

(paper: they use specializations)

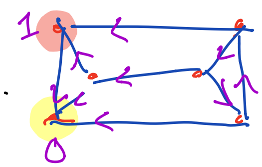
Informally: Factors of C_G reveal information about rigid substructures [and much more]

Application 2: Combinatorial formulas

Defn: A tie-down of (G,p) drops $\binom{d+1}{2}$ colums from $R(p)$ s.t. rank is still $dn - \binom{d+1}{2}$.

A compatible d-fan is an orientation of G s.t.

$$\forall i \in V \quad \text{out-deg}(i) = d - \#\text{tied down cols.}$$



$$\begin{bmatrix} R(p) \\ \hline Id \end{bmatrix}$$

extra rows are a tie down

Then

equality in bracket ring

edge vectors

$$C_G = \pm \sum_{d\text{-fans}} \prod_{i} [e_{i i_1} \cdots e_{i i_d}]$$

↳ edges oriented out of i

need this b/c o/w my choice of tie down chooses things

give polynomials in coords

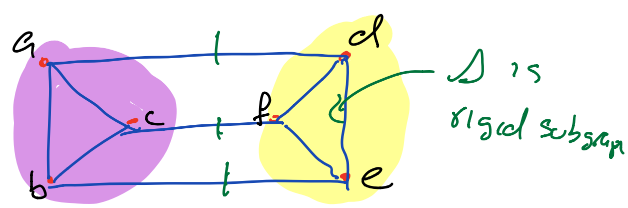
Tree formulas : [Whiteley '96], [Streinu-LT '10],

More general constraints : [Sidman - St John '13] ← "body CAD constraints"

Lots of questions:

cf [Sturmfels - Whiteley JSC '93]

- Cayley factorization
  ↳ expressing everything in terms of spans and intersections
- Geometry of all factors of C_G



green lines concurrent

$$[abc][def]\left([abe][dcf] - [dbe][afc]\right)$$

- What about graph with more edges? [Bolker-Roth] complete bipartite graphs
- Jessica Sidman: Relationship between coset reps and pebble game?

- can you enumerate all the d-fans using pebble game moves?
- Answer: Whitely and collaborators, "Assur decompositions" Shai - Servatius - Whiteley.

Advertisements:

https://www.researchgate.net/publication/244505791_The_Algebraic_Geometry_of_Stresses_in_Frameworks

get the paper!

http://www.fields.utoronto.ca/activities/20-21/constraint

upcoming program on rigidity

Conj / Questions (Walter):

In 2d if C_G factors, is there a proper rigid subgraph?

If C_G has a factor f^k $k \geq 2$, and $f(p) = 0$, does (G, p) have eq. stress space dim > 1? (hardér $= k$)

Ex: $K_{4,6}$ in 3d has a deg 2 factor

$$[p_1 \cdots p_4]^2$$

does have 2 stresses (e.g. by Bolker-Roth).

Comment from Walter: Precursor:

Assor's work in c. 1910.