jt w/ Rosen, Theran, Vinzant

- $G = (V, E)$, $|V| = n$
- $p = (p_1, \dots, p_n) \in (\mathbb{R}^d)^n$
- $G(p)$ = framework w/ ith joint at p_i

Q: Is $G(p)$ rigid or flexible?

Ex: 

Thm (Asimow–Roth) Given G, almost all $G(p)$ are rigid, or almost all are flexible.

Q: If G is gen. minimally rigid, for which $G(p)$ does genericity fail?

$x = (x_1, \dots, x_n)$ $\qquad (l_{ij})$ sq. dist.

$$\begin{tikzcd}
\varphi : (\mathbb{R}^d)^n \arrow[rr] \arrow[dr, "\varphi_G"'] & & \mathbb{R}^{\binom{n}{2}} \arrow[dl, "\pi_G"] \\
& \mathbb{R}^{|E|} &
\end{tikzcd}$$

$\varphi_G^{-1}(l)$ = solns of

$$(x_i - x_j)\cdot(x_j - x_i) = l_{ij} \quad \forall ij \in E$$

Infinitesimal setting:

$d\psi_G$ = rigidity matrix w/ $|E|$ rows

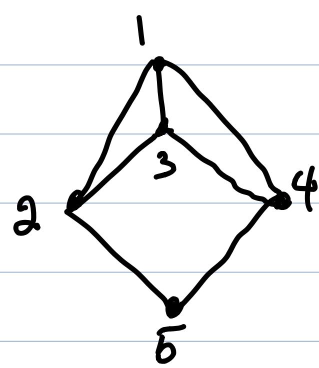
$$\text{row } ij: \quad \underbrace{\begin{array}{ccccccc} 1 & \cdots & i & \cdots & j & \cdots & n \\ \vec{0} & & x_i - x_j & & x_j - x_i & & \vec{0} \end{array}}_{dn \text{ cols}}$$

- stress := dep. on rows
- infinitesimal motion := elt of kernel
- rigidity matroid: G is indep if $d\psi_G$ has lin. ind. rows.

<u>White–Whiteley</u>: Describe p s.t. $(d\psi_G)_p$ drops rank. (G gen. min. rigid)

- as $C_G(x) = 0$
- geometry of p can be described using synthetic geom. conditions (sometimes)

<u>Ex</u>: $n=5$, $d=2$



$C_G(x) = [123][134][245]$

$$[123] = \det\begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{bmatrix}$$

$[123] = 0 \implies x_1, x_2, x_3$ are colin.
rows 12, 13, 23 are dep.

$[245] = 0 \implies x_2, x_4, x_5$ are colin.
dep.... supported on all of G.

<u>note</u>: Information about edges obscured.

<u>Q</u>: What if we work with (l_{ij})?

$$\begin{tikzcd} \overset{x}{(\mathbb{C}^d)^n} \arrow[r, "\varphi"] \arrow[dr, "\varphi_G"'] & \overset{l}{\mathbb{C}^{\binom{n}{2}}} \arrow[d, "\pi_G"] \\ & \mathbb{C}^{|E|} \end{tikzcd} \qquad \varphi^*: \mathbb{C}[l_{ij}] \to \mathbb{C}[x]$$

$V_{d,n}$: $\overline{\varphi(\mathbb{C}^{dn})}$ = Cayley-Menger variety
(Borcea, Streinu)

$I_{d,n}$ = all polys van. on $V_{d,n}$
= algebraic relns on l_{ij}

We can recover the rigidity matroid from $I_{d,n}$:
$G = (V, E)$, $|V| = n$

$$I_G := I_{d,n} \cap \mathbb{C}[l_{ij} \mid ij \in E]$$

$$I_G = \begin{cases} \langle 0 \rangle & G \text{ is ind.} \\ \neq \langle 0 \rangle & G \text{ is dep.} \\ \langle \text{irred. poly} \rangle & G \text{ is a circuit} \end{cases}$$

$\text{irred. poly} = f_G$ = circuit poly. if support of $f_G = G$

Q: How can we use elts of I_G to understand non-generic frameworks?

- supports of stresses
- motions

Obs: $f \in I_G \Rightarrow \nabla f \cdot (d\varphi_G) = \vec{0}$

$f(\varphi(p)) = 0$

∇f: coeffs of a stress

$d\varphi_G$: rigidity matrix

Thm: (S, R, T, V) G min. rigid.

- g irred factor of $C_G(x)$
- $(\varphi^*)^{-1}(\langle g \rangle)$ is prime

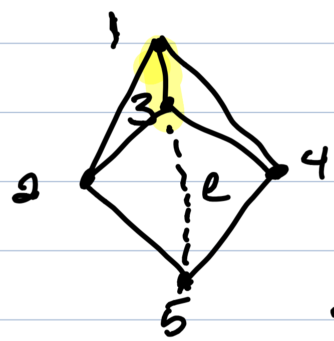
$\Rightarrow P = (\varphi^*)^{-1}(\langle g \rangle) \cap \mathbb{C}[l_{ij} \mid ij \in G] \neq \langle 0 \rangle$

$\forall f \in P$, ∇f is a stress on $G(p)$ if $p \in V(g)$

note: If $P = \langle f \rangle$, then f is a factor of the discriminant of φ_G.

Q: If G is min. rigid, for which (l_{ij}) can motions occur?

Ex:



$n=5$, $d=2$

Is there a motion for which $e=35$ changes length?

$G' = G + e$ is a circuit

$f_{G'}$ circuit poly.

$$f_{G'}(e) = f_6 e^6 + \cdots + f_1 e + f_0$$

If (l_{ij}) allows a motion changing the length of e, then $(l_{ij}) \in V(f_0, \ldots, f_6)$.

$\langle f_0, \ldots, f_6 \rangle$ has 6 assoc. primes.

two: $\langle l_{13}, l_{14} - l_{34}, l_{12} - l_{23} \rangle$

$\langle l_{23} - l_{24},\ l_{12} - l_{14},\ l_{25} - l_{45} \rangle$