Epsilon local rigidity and numerical algebraic geometry

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- Connected graph (V, E) with n nodes and m edges.
- Embedded in $\mathbb{R}^d$ by the map $p : [n] \to \mathbb{R}^d$
- We say $p = (p_{ik}) \in \mathbb{R}^{nd}$.
- A *deformation* of p is a continuous map $p(t) : [0, 1] \to \mathbb{R}^{nd}$.
- A *rigid motion* is a deformation preserving

$$\left\{ \sum_{k=1}^{d} (p_{ik}(t) - p_{jk}(t))^2 \right\}_{ij \in \binom{[n]}{2}}$$

- Euclidean group of rigid motions has dimension $\binom{d+1}{2}$.

Consider $p : [n] \to \mathbb{R}^d$, and $E \subset \binom{[n]}{2}$, $|E| = m$.

$$[6] = \{1, 2, 3, 4, 5, 6\}$$
$$E = \{12, 13, 14, 15, 23, 25, 26, 34, 36, 45, 46, 56\}$$

$$p = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \\ p_{41} & p_{42} & p_{43} \\ p_{51} & p_{52} & p_{53} \\ p_{61} & p_{62} & p_{63} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 3 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 3 \\ 0 & 1 & 3 \end{bmatrix}.$$

We say $p \in \mathbb{R}^{nd}$.

$$
\begin{array}{c} p \in \mathbb{R}^{nd} \\ E \subset \binom{[n]}{2} \\ |E| = m \end{array} \quad \Longrightarrow \quad \begin{array}{c} g : \mathbb{C}^{nd} \to \mathbb{C}^{m} \\ g(x) = 0 \\ V(g), V_{\mathbb{R}}(g) \end{array}
$$

$$
V(g) := \left\{ x \in \mathbb{C}^{nd} \ : \ g(x) = 0 \right\}
$$

$$
V_{\mathbb{R}}(g) := V(g) \cap \mathbb{R}^{nd}.
$$

$$\begin{array}{c} p \in \mathbb{R}^{nd} \\ E \subset \binom{[n]}{2} \\ |E| = m \end{array} \quad \Longrightarrow \quad \begin{array}{c} g : \mathbb{C}^{nd} \to \mathbb{C}^m \\ g(x) = 0 \\ V(g), V_{\mathbb{R}}(g) \end{array}$$

$$g(x) = \begin{bmatrix} (x_{11}-x_{21})^2 + (x_{12}-x_{22})^2 + (x_{13}-x_{23})^2 - 3 \\ (x_{11}-x_{31})^2 + (x_{12}-x_{32})^2 + (x_{13}-x_{33})^2 - 3 \\ (x_{11}-x_{41})^2 + (x_{12}-x_{42})^2 + (x_{13}-x_{43})^2 - \frac{1}{4}\left(\sqrt{3}+2\right)^2 - \frac{37}{4} \\ (x_{11}-x_{51})^2 + (x_{12}-x_{52})^2 + (x_{13}-x_{53})^2 - \frac{1}{4}\left(\sqrt{3}-2\right)^2 - \frac{37}{4} \\ (x_{21}-x_{31})^2 + (x_{22}-x_{32})^2 + (x_{23}-x_{33})^2 - 3 \\ (x_{21}-x_{51})^2 + (x_{22}-x_{52})^2 + (x_{23}-x_{53})^2 - \frac{1}{2}\left(\sqrt{3}+1\right)^2 - 9 \\ (x_{21}-x_{61})^2 + (x_{22}-x_{62})^2 + (x_{23}-x_{63})^2 - \frac{1}{4}\left(\sqrt{3}-2\right)^2 - \frac{37}{4} \\ (x_{31}-x_{41})^2 + (x_{32}-x_{42})^2 + (x_{33}-x_{43})^2 - \frac{1}{2}\left(\sqrt{3}-1\right)^2 - 9 \\ (x_{31}-x_{61})^2 + (x_{32}-x_{62})^2 + (x_{33}-x_{63})^2 - \frac{1}{4}\left(\sqrt{3}+2\right)^2 - \frac{37}{4} \\ (x_{41}-x_{51})^2 + (x_{42}-x_{52})^2 + (x_{43}-x_{53})^2 - 3 \\ (x_{41}-x_{61})^2 + (x_{42}-x_{62})^2 + (x_{43}-x_{63})^2 - 3 \\ (x_{51}-x_{61})^2 + (x_{52}-x_{62})^2 + (x_{53}-x_{63})^2 - 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Definition

A *flex* of p is a deformation $p(t) : [0, 1] \to \mathbb{R}^{nd}$ such that $g(p(t)) = 0$ for all $t \in [0, 1]$ and which is not a rigid motion.

Definition

The configuration p is called *locally rigid* if no flex exists.

In the 2009 paper by Timothy Abbott, Reid Barton, and Eric Demaine, "Generalizations of Kempe's Universality Theorem" [ABD09] deciding local rigidity was shown to be Co-NP hard.

$$\begin{array}{c} p \in \mathbb{R}^{nd} \\ E \subset \binom{[n]}{2} \\ |E| = m \end{array} \quad \Longrightarrow \quad \begin{array}{c} g : \mathbb{C}^{nd} \to \mathbb{C}^m \\ g(x) = 0 \\ V(g), V_{\mathbb{R}}(g) \end{array}$$

The Jacobian $dg : \mathbb{C}^{nd} \to \mathbb{C}^m$ is a linear map, a matrix.

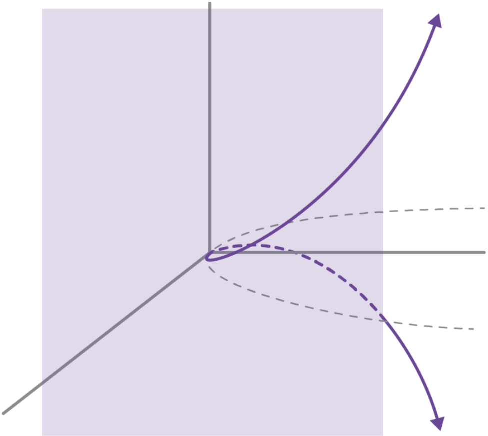
Jacobian $dg : \mathbb{C}^{nd} \to \mathbb{C}^m$ is a linear map, a matrix.

$$\frac{1}{2}dg = \left(\begin{array}{ccccccc}
x_{11}-x_{21} & x_{12}-x_{22} & x_{13}-x_{23} & -x_{11}+x_{21} & -x_{12}+x_{22} & -x_{13}+x_{23} & 0 \\
x_{11}-x_{31} & x_{12}-x_{32} & x_{13}-x_{33} & 0 & 0 & 0 & -x_{11}+x_{31} \\
x_{11}-x_{41} & x_{12}-x_{42} & x_{13}-x_{43} & 0 & 0 & 0 & 0 \\
x_{11}-x_{51} & x_{12}-x_{52} & x_{13}-x_{53} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_{21}-x_{31} & x_{22}-x_{32} & x_{23}-x_{33} & -x_{21}+x_{31} \\
0 & 0 & 0 & x_{21}-x_{51} & x_{22}-x_{52} & x_{23}-x_{53} & 0 \\
0 & 0 & 0 & x_{21}-x_{61} & x_{22}-x_{62} & x_{23}-x_{63} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_{31}-x_{41} \\
0 & 0 & 0 & 0 & 0 & 0 & x_{31}-x_{61} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right. \cdots$$

$$\cdots \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
-x_{12}+x_{32} & -x_{13}+x_{33} & 0 & 0 & 0 & 0 \\
0 & 0 & -x_{11}+x_{41} & -x_{12}+x_{42} & -x_{13}+x_{43} & 0 \\
0 & 0 & 0 & 0 & 0 & -x_{11}+x_{51} \\
-x_{22}+x_{32} & -x_{23}+x_{33} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -x_{21}+x_{51} \\
0 & 0 & 0 & 0 & 0 & 0 \\
x_{32}-x_{42} & x_{33}-x_{43} & -x_{31}+x_{41} & -x_{32}+x_{42} & -x_{33}+x_{43} & 0 \\
x_{32}-x_{62} & x_{33}-x_{63} & 0 & 0 & 0 & 0 \\
0 & 0 & x_{41}-x_{51} & x_{42}-x_{52} & x_{43}-x_{53} & -x_{41}+x_{51} \\
0 & 0 & x_{41}-x_{61} & x_{42}-x_{62} & x_{43}-x_{63} & 0 \\
0 & 0 & 0 & 0 & 0 & x_{51}-x_{61}
\end{array} \cdots$$

$$\cdots \left.\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-x_{12}+x_{52} & -x_{13}+x_{53} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-x_{22}+x_{52} & -x_{23}+x_{53} & 0 & 0 & 0 \\
0 & 0 & -x_{21}+x_{61} & -x_{22}+x_{62} & -x_{23}+x_{63} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -x_{31}+x_{61} & -x_{32}+x_{62} & -x_{33}+x_{63} \\
-x_{42}+x_{52} & -x_{43}+x_{53} & 0 & 0 & 0 \\
0 & 0 & -x_{41}+x_{61} & -x_{42}+x_{62} & -x_{43}+x_{63} \\
x_{52}-x_{62} & x_{53}-x_{63} & -x_{51}+x_{61} & -x_{52}+x_{62} & -x_{53}+x_{63}
\end{array}\right)$$

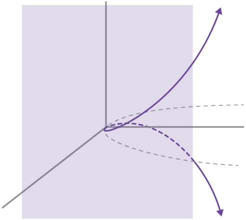
$$f(x) = \begin{bmatrix} f_1 = -x^3 + xy \\ f_2 = -x^4 + xz \\ f_3 = x^7 - x^5y - x^4z + x^2yz \end{bmatrix} : \mathbb{C}^3 \to \mathbb{C}^3$$



$$df = \begin{bmatrix} -3x^2 + y & x & 0 \\ -4x^3 + z & 0 & x \\ 7x^6 - 5x^4 y - 4x^3 z + 2xyz & -x^5 + x^2 z & -x^4 + x^2 y \end{bmatrix}$$

$df|_{(0,0,0)}$ $\quad$ $df|_{(0,5,3)}$ $\quad$ $df|_{(1,1,1)}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$



Jacobian $dg : \mathbb{C}^{nd} \to \mathbb{C}^m$ has a *generic rank*. Choosing $q = (q_{ik}) \in \mathbb{C}^{nd}$ randomly, $dg|_q$ becomes a matrix with scalar entries. Calculate its rank (Gaussian elimination if exact computation, or SVD for floating point calculations).

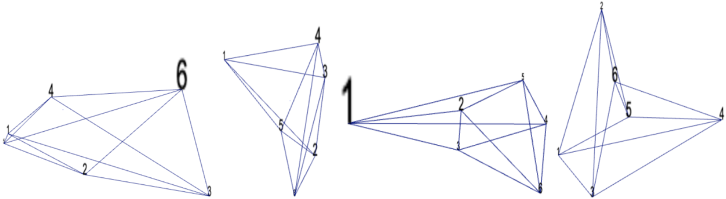


Figure 5: Some random 3-prisms in $\mathbb{R}^3$

By the way, $3/4$ of triangles are obtuse. From Edelman and Strang:

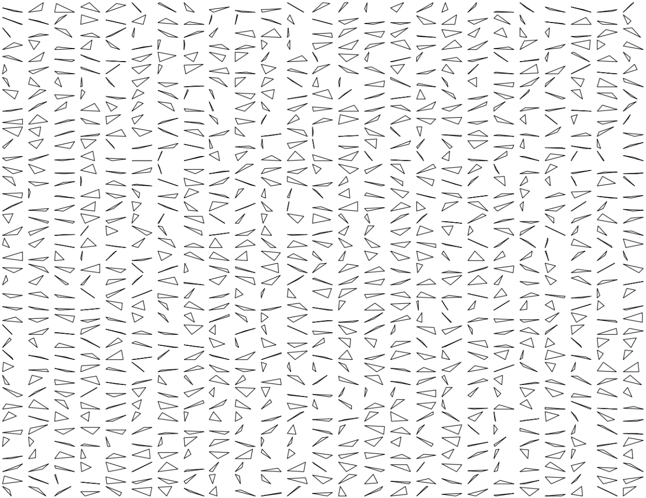


Figure 3: 1000 Random Triangles (Gaussian Distribution): Most triangles are obtuse.

Theorem

A configuration p with n nodes embedded in $\mathbb{R}^d$ is infinitesimally rigid if

$$\mathrm{rank}(dg|_p) = nd - \binom{d+1}{2},$$
$$\mathrm{corank}(dg|_p) = \binom{d+1}{2}.$$

Theorem

Infinitesimal rigidity implies local rigidity.

Why does this work?

Theorem

Infinitesimal rigidity implies local rigidity.

Proof.

Since the Euclidean group acts we have a lower bound

$$\binom{d+1}{2} \leq \mathsf{dim}_p V_{\mathbb{R}}(g).$$

But

$$\mathsf{dim}_p V_{\mathbb{R}}(g) \leq \mathsf{dim}_p V(g) \leq \mathsf{corank}(dg|_p) = \binom{d+1}{2}.$$

$\square$

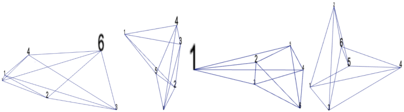


Figure 5: Some random 3-prisms in $\mathbb{R}^3$

$$p = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 3 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

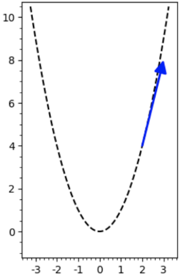


Figure 12: Tangent space

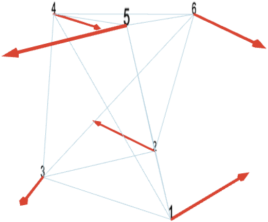


Figure 13: Infinitesimal flexes

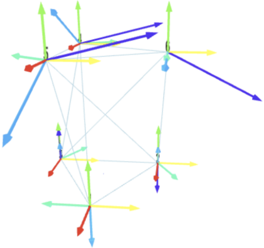
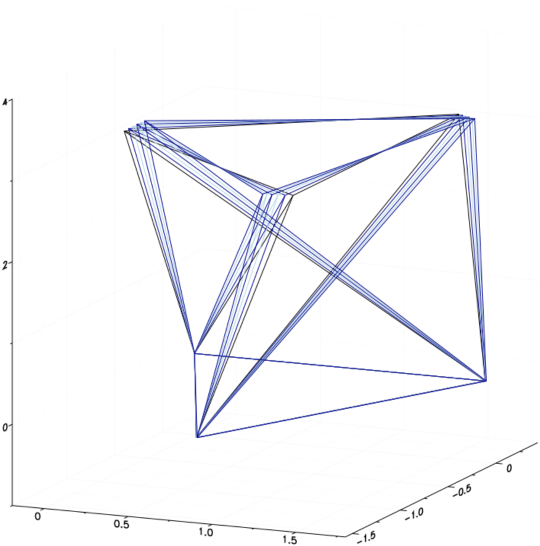
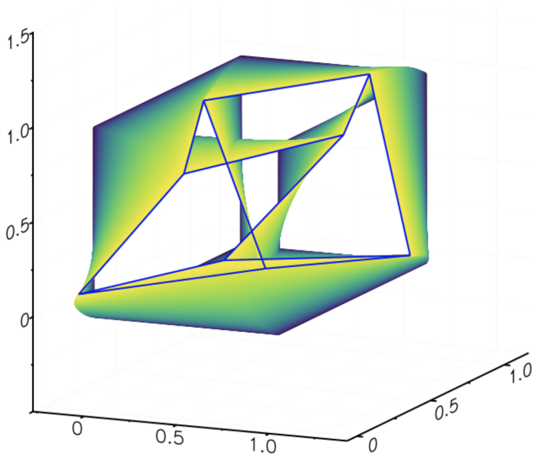


Figure 14: Infinitesimal rigid motions

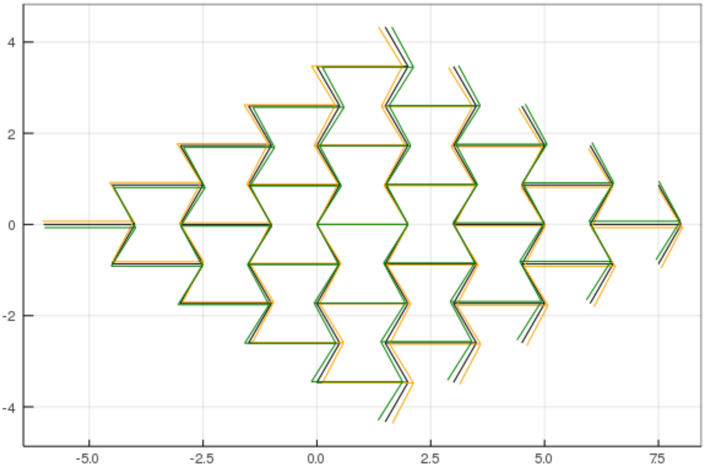
Flexible?



Flexible?

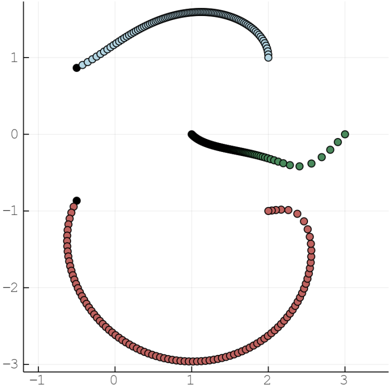


Flexible?

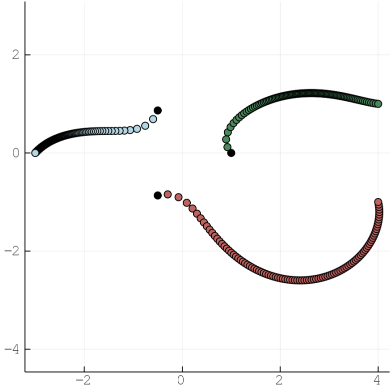


…and now for something completely different.

We know the roots of $q(x) = x^3 - 1$.



$$x^3 - 7x^2 + 17x - 15$$
$$(x-3)(x-2-i)(x-2+i)$$

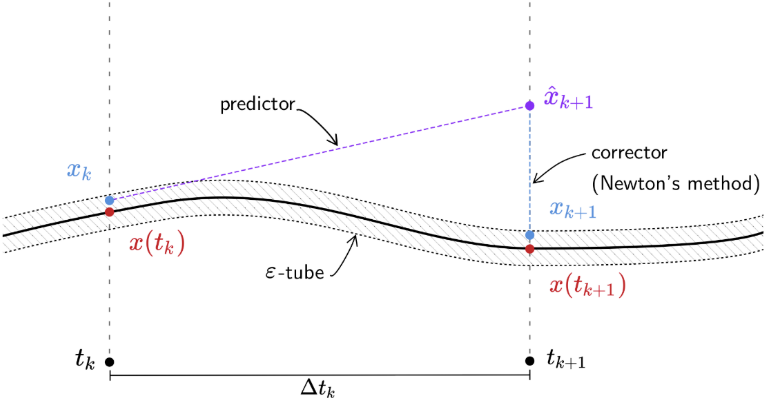


$$x^3 - 5x^2 - 7x + 51$$
$$(x+3)(x-4+i)(x-4-i)$$

$$h(z,t) = (1-t)\begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_N(x) \end{bmatrix} + \gamma t \begin{bmatrix} x_1^{d_1} - 1 \\ x_2^{d_2} - 1 \\ \vdots \\ x_N^{d_N} - 1 \end{bmatrix},$$

$$\begin{aligned} h(x,t) &= (1-t)f + \gamma t q \\ h(x(t),t) &= 0 \end{aligned}$$

$$h(x,t) = (1-t)f + \gamma t q$$
$$h(x(t),t) = 0$$
$$\frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial t} = 0.$$



Say that $f : \mathbb{C}^7 \to \mathbb{C}^4$. Its irreducible components X can have possible dimensions dim $X \in \{3, 4, 5, 6\}$. To find 4-dimensional components, create a square system of equations $\mathbb{C}^7 \to \mathbb{C}^7$:

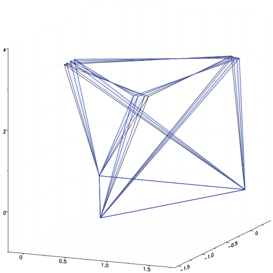
$$\begin{bmatrix} Af \\ Lx \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 & c_{11} \\ 0 & 1 & 0 & c_{21} \\ 0 & 0 & 1 & c_{31} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \\ \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{17} \\ L_{21} & L_{22} & \cdots & L_{27} \\ L_{31} & L_{32} & \cdots & L_{37} \\ L_{41} & L_{42} & \cdots & L_{47} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

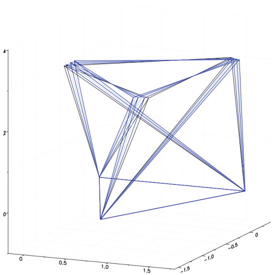
The numerical irreducible decomposition uses *witness sets* and the following

Theorem (Bertini's Theorem, Theorem 9.3 of [BSHW13])

Given a polynomial system $f : \mathbb{C}^N \to \mathbb{C}^n$*, there is a Zariski-open, dense set* $U \subset \mathbb{C}^{k\times n}$ *of matrices* A *such that* $V(Af) \setminus V(f)$ *is either empty or consists of exactly* $C_f \in \mathbb{Z}_{>0}$ *irreducible components, each smooth (and hence disjoint) and of dimension* $N - k$*. The number* C_f *of these extraneous components is independent of* A*.*

But now back to epsilon local rigidity...





Using a *moving frame* [Olv], we can change coordinates in a useful way, creating zeros.

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 3 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 3 \\ 0 & 1 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 1.7320508075688772 & 0.0 & 0.0 \\ 0.8660254037844388 & -1.5 & 0.0 \\ 1.3660254037844386 & -1.3660254037844386 & 3.0 \\ -0.1339745962155613 & -0.5 & 3.0 \\ 1.3660254037844388 & 0.3660254037844386 & 3.0 \end{bmatrix}.$$

Real parameter homotopy

Start with equations $g(x) = 0$ but then adjoin one additional equation:

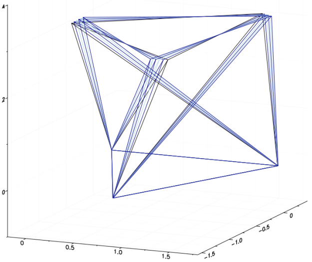
$$\ell_v := v^T x - v^T p = 0.$$

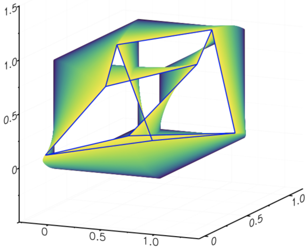
Here, we can choose $v \in \mathbb{R}^N$ randomly, or we could choose v from some infinitesimal flex. Then *perturb* this equation to

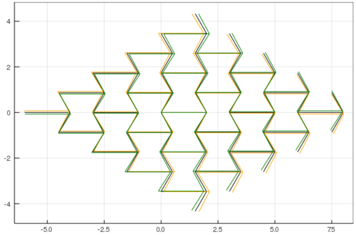
$$\ell_{v,\epsilon} := v^T x - v^T p - \epsilon = 0.$$

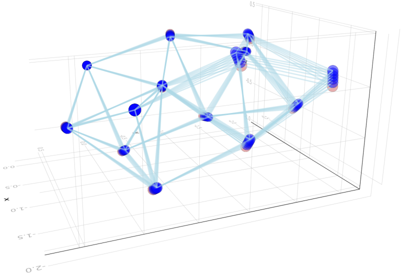
for some small real $0 < \epsilon \in \mathbb{R}$. In the computer, we use a *real parameter homotopy* (without γ) as in

$$h(x,t) = (1-t)\begin{bmatrix} g \\ \ell_{v,\epsilon} \end{bmatrix} + t \begin{bmatrix} g \\ \ell_v \end{bmatrix}.$$

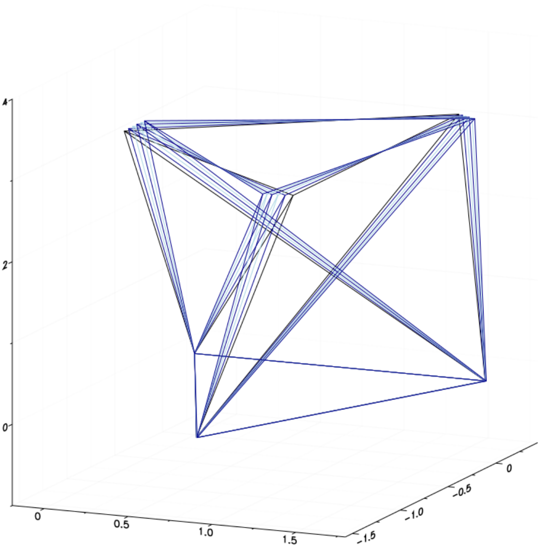

2
0
0
0.5
1.0
1.5
0
-0.5
-1.0
-1.5


1.5
1.0
0.5
0
0
0.5
1.0
0
0.5
1.0


4
2
0
-2
-4
-5.0
-2.5
0.0
2.5
5.0



Flexible?



Flexible?

Here is one of the first deformations where the 3-prism begins to twist downwards, as you can see in the z-coordinates of nodes $4, 5, 6$.

$$\begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 1.7345015098619578 & 0.0 & 0.0 \\ 0.868440136662386 & -1.4992275136004456 & 0.0 \\ 1.434394418877309 & -1.322820159782012 & 2.9867578031247515 \\ -0.12780675639331027 & -0.5722571170386941 & 2.9884237516052568 \\ 1.3032879403929745 & 0.40433597324593706 & 2.9865056925919635 \end{bmatrix}$$

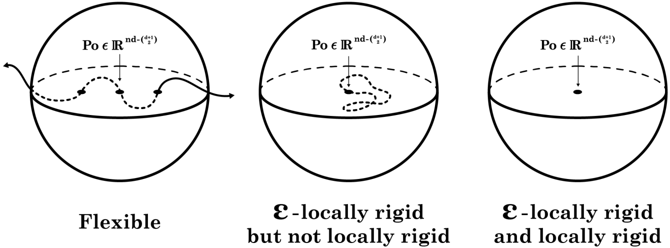
The 3-prism can also untwist upwards, as you can see the configuration below:

$$\begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 1.7366579715554198 & 0.0 & 0.0 \\ 0.8689191994267838 & -1.5000751141689224 & 0.0 \\ 1.2441020030189796 & -1.4251233353656827 & 3.022965523609326 \\ -0.12034129224092607 & -0.35340935623990566 & 3.024004212778408 \\ 1.4887072950829638 & 0.2911265336721315 & 3.021803451042337 \end{bmatrix}$$

Epsilon local rigidity

Definition

Let p_0 be an initial configuration and $\widehat{p_0}$ be the configuration in the moving frame. We say that p_0 is ε-*locally rigid* if every flex $\widehat{p(t)}$ of $\widehat{p_0}$ satisfies $\widehat{p(t)} \in B_\varepsilon(\widehat{p_0})$ for all $t \in [0,1]$, where $B_\varepsilon(\widehat{p_0})$ is the open ε-ball centered at $\widehat{p_0}$.



The nonlinear system of equations

We note this draws on results from [ARSED02, RRSED00] and also from the 1954 paper of Seidenberg [Sei54].

Theorem (Theorem 5 of [Hau12])

Suppose that the conditions in the Assumption hold. Let $z \in \mathbb{R}^{N-k}$, $\gamma \in \mathbb{C}$, $y \in \mathbb{R}^N - V_{\mathbb{R}}(f)$, $\alpha \in \mathbb{C}^{N-k+1}$, *and* $H : \mathbb{C}^N \times \mathbb{C}^{N-k+1} \times \mathbb{C} \to \mathbb{C}^{2N-k+1}$ *be the homotopy defined by*

$$H(x, \lambda, t) = \left[\begin{array}{c} f(x) - t\gamma z \\ \lambda_0(x - y) + \lambda_1 \nabla f_1(x)^T + \cdots + \lambda_{N-k} \nabla f_{N-k}(x)^T \\ \alpha^T \lambda - 1 \end{array} \right]$$

where $f(x) = [f_1(x), \ldots, f_{N-k}(x)]^T$. *Then*

$$E_1 \cap V \cap \mathbb{R}^N$$

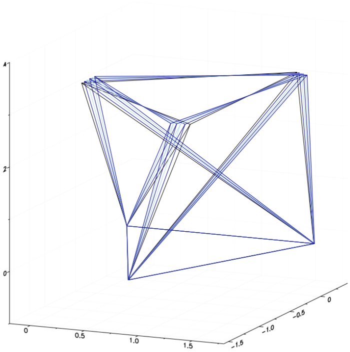
contains a point on each connected component of $V_{\mathbb{R}}(f)$ *contained in* V.

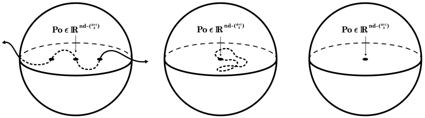
The nonlinear system of equations

We collect here the following list of assumptions which refer to the homotopy $H(x, \lambda, t)$ defined above.

1. Let $N > k > 0$ and $f : \mathbb{R}^N \to \mathbb{R}^{N-k}$ be a polynomial system with real coefficients, with $V \subset V(f)$ a pure k-dimensional algebraic set with witness set $\{f, L, W\}$.
2. Assume that the starting solutions to $H(x, \lambda, 1) = 0$ are finite and nonsingular.
3. Assume also that the number of starting solutions is equal to the maximum number of isolated solutions to $H(x, \lambda, 1) = 0$ as z, γ, y, α vary over $\mathbb{C}^{N-k} \times \mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^{N-k+1}$. This will be true for a nonempty Zariski open set of $\mathbb{C}^{N-k} \times \mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^{N-k+1}$.
4. Assume all the solution paths defined by H starting at $t = 1$ are trackable. This means that for each starting solution (x^*, λ^*) there exists a smooth map $\xi : (0, 1] \to \mathbb{C}^N \times \mathbb{C}^{N-k+1}$ with $\xi(1) = (x^*, \lambda^*)$ and for all $t \in (0, 1]$ we have $\xi(t)$ is a nonsingular solution of $H(x, \lambda, t)$.
5. Assume that each solution path converges, collecting the endpoints of all solution paths in the sets E and $E_1 = \pi(E)$ where $\pi(x, \lambda) = x$ projects onto the x coordinates, forgetting the λ coordinates.

The 3-prism is epsilon locally rigid.



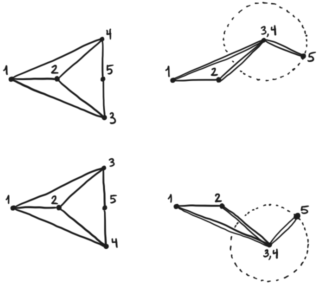


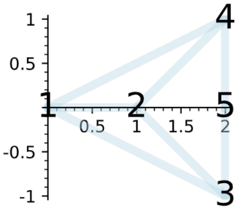
Flexible

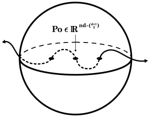
$\mathcal{E}$-locally rigid but not locally rigid

$\mathcal{E}$-locally rigid and locally rigid

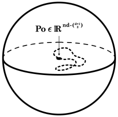
The slingshot is epsilon locally rigid.



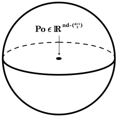




Flexible



ε-locally rigid but not locally rigid



ε-locally rigid and locally rigid

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Prestress rigidity tells us the 3-prism is rigid.

$$\begin{aligned} A &:= \text{incidence matrix of the graph} \\ w^T(dg|_p) &= 0 \\ K_c &= (dg|_p)^T \mathsf{diag}(c)(dg|_p) \\ H_{w,c}(x) &= x^T \left(\Omega_w + K_c\right) x \\ \Omega_w &= A^T \mathsf{diag}(w) A \otimes I_d \\ F^T \left(a_1 \cdot \Omega_{w_1} + a_2 \cdot \Omega_{w_2}\right) F &\succ 0 \\ v_1^T \left(a_1 \cdot \Omega_{w_1}\right) v_1 &> 0 \\ v^T \Omega_w v &= 89.56922 > 0 \end{aligned}$$