Flexible circuits and d-dimensional rigidity

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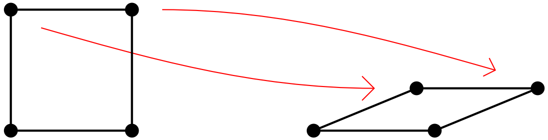
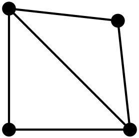
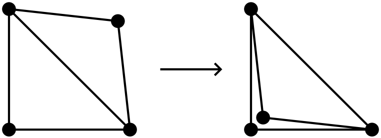
Rigidity

- A bar-joint framework (G, p) is the combination of a graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$.
- When do the lengths (locally) determine the shape?
- A framework (G, p) is (continuously) rigid if every edge-length preserving continuous motion of the vertices of (G, p) arises from an isometry of $\mathbb{R}^d$.





- This is rigid in 2D but has other realisations.

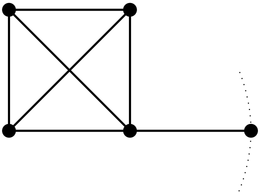


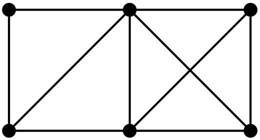
- A framework (G, p) is globally rigid if every framework (G, q) with the same edge lengths as (G, p) arises from an isometry of $\mathbb{R}^d$.
- This talk will focus on rigidity.

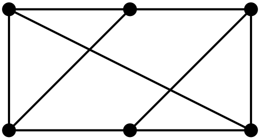
- For frameworks on the real line, everything is simple:
- Folklore: A framework (G, p) is rigid in $\mathbb{R}$ if and only if G is connected.

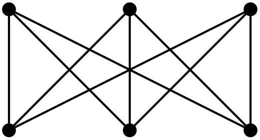
- In dimension greater than 1 it is NP-hard to determine if a given framework is rigid (Abbott 2008).

Examples - in the plane









A linearisation

- An infinitesimal motion of a framework (G, p) is a map $s : V \to \mathbb{R}^d$ such that $(p_j - p_i) \cdot (s_j - s_i) = 0$ for all $v_j v_i \in E$.
- The rigidity matrix is the $|E| \times d|V|$ matrix $R(G, p)$ whose rows are indexed by E and d-tuples of columns indexed by V in which, for $e = v_i v_j \in E$, the row has the form:

$$\begin{pmatrix} \dots & p_i - p_j & \dots & p_j - p_i & \dots \end{pmatrix}.$$

- (G, p) is infinitesimally rigid if every infinitesimal motion is an infinitesimal isometry of $\mathbb{R}^d$, or equivalently if the rigidity matrix has rank $d|V| - \binom{d+1}{2}$.
- The rigidity matrix gives rise to the generic d-dimensional rigidity matroid $\mathcal{R}_d$.
- (G, p) is $\mathcal{R}_d$-independent if $R(G, p)$ has linearly independent rows.

Asimow and Roth

- A framework (G, p) is generic if the coordinates of p form an algebraically independent set over $\mathbb{Q}$.

Theorem: Asimow and Roth 1978

Let (G, p) be a generic framework in $\mathbb{R}^d$. Then (G, p) is rigid if and only if it is infinitesimally rigid.

- Hence, generically, rigidity is a property of the graph in every dimension.
- We say a graph G is $\mathcal{R}_d$-rigid if some (and hence every) generic framework (G, p) is rigid.

Maxwell's necessary conditions

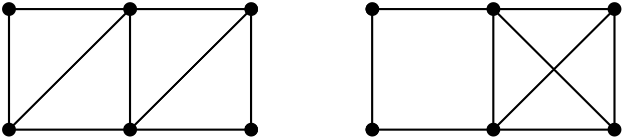
- A graph $G = (V, E)$ is $(d, \binom{d+1}{2})$-tight if $|E| = d|V| - \binom{d+1}{2}$ and for any subgraph (V', E'), with $|V'| \geq d$, we have $|E'| \leq d|V'| - \binom{d+1}{2}$.

Lemma - Maxwell 1864

Let $G = (V, E)$ be $\mathcal{R}_d$-rigid with $|V| \geq d + 1$. Then G contains a spanning subgraph H that is $(d, \binom{d+1}{2})$-tight.

- A major problem in rigidity theory is to establish sufficient combinatorial conditions for a graph to be rigid.

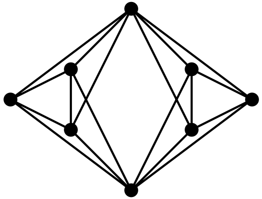
- A graph $G = (V, E)$ is (2, 3)-tight if $|E| = 2|V| - 3$ and for any subgraph (V', E') with $|V'| \geq 2$ we have $|E'| \leq 2|V'| - 3$.



Theorem: Laman 1970, Pollaczek-Geiringer 1927

A graph G is $\mathcal{R}_2$-rigid if and only if G contains a spanning subgraph that is $(2, 3)$-tight.

- The converse fails in all dimensions $d \geq 3$.
- For example, here is a $(3,6)$-tight graph that is flexible in $\mathbb{R}^3$.

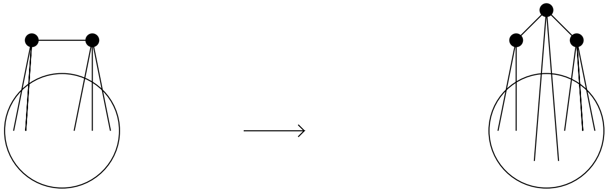


Partial results

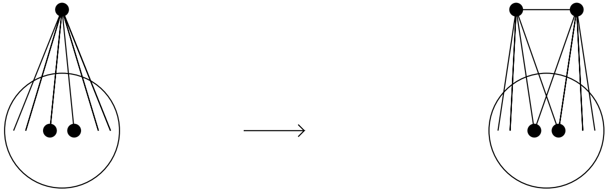
- There are a number of partial results, I'll mention just a few.
- Complete bipartite graphs - Bolker and Roth 1980.
- Triangulations - Cauchy 1813, Dehn 1916, Gluck 1975, Fogelsanger 1988.
- Molecular frameworks - Katoh and Tanigawa 2011.
- Abstract rigidity/cofactor matroids - Sitharam and Vince 2015+, Clinch, Jackson and Tanigawa 2019.

Graph operations

- We will need several standard graph operations.
- A graph G' is said to be obtained from another graph G by: a *0-extension* if $G = G' - v$ for a vertex $v \in V(G')$ with $d_{G'}(v) = d$; or a *1-extension* if $G = G' - v + xy$ for a vertex $v \in V(G')$ with $d_{G'}(v) = d + 1$ and $x, y \in N(v)$.



- A *vertex split* of a graph $G = (V, E)$ is defined as follows: choose $v \in V$, $x_1, x_2, \ldots, x_{d-1} \in N(v)$ and a partition N_1, N_2 of $N(v) \setminus \{x_1, x_2, \ldots, x_{d-1}\}$; then delete v from G and add two new vertices v_1, v_2 joined to N_1, N_2, respectively; finally add new edges $v_1v_2, v_1x_1, v_2x_1, v_1x_2, v_2x_2, \ldots, v_1x_{d-1}, v_2x_{d-1}$.



Lemma

Let G be $\mathcal{R}_d$-independent and let G' be obtained from G by a 0-extension or a 1-extension. Then G' is $\mathcal{R}_d$-independent.

Theorem - Whiteley 1990

Let G be $\mathcal{R}_d$-independent and let G' be obtained from G by a vertex split. Then G' is $\mathcal{R}_d$-independent.

Theorem - Whiteley 1983

Let $d \geq 1$ be an integer, G be a graph and let G' be obtained from G by adding a new vertex adjacent to every vertex of G. Then G is $\mathcal{R}_d$-independent if and only if G' is $\mathcal{R}_{d+1}$-independent.

Open problem - X-replacement

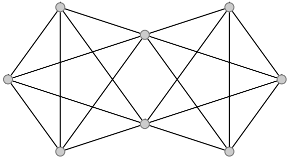
- An X-replacement removes two non-adjacent edges xy, zw and adds a degree $d+2$ vertex v adjacent to x, y, z, w and $d-2$ additional distinct vertices.
- Conjectured to preserve 3-dimensional rigidity - Graver, Tay and Whiteley 1980s.
- Easy proof in dimension 2. Known that it sometimes fails to preserve independence in dimension ≥ 4.
- Some special cases in 3D are known - e.g. Cruickshank 2014.

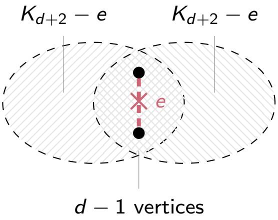
Flexible circuits

- Recall $\mathcal{R}_d$ denotes the row matroid of the rigidity matrix $R(G, p)$ for generic (G, p) in $\mathbb{R}^d$.
- We study $\mathcal{R}_d$-circuits - graphs whose rigidity matrix has a minimally dependent set of rows.
- In dimensions 1 and 2 every $\mathcal{R}_d$-circuit is rigid (this follows, e.g., from Laman's theorem).
- The double banana shows that $\mathcal{R}_d$-circuits can be flexible when $d \geq 3$.
- We want to understand the structure of flexible $\mathcal{R}_d$-circuits.
- Assume $d \geq 3$ from here on.

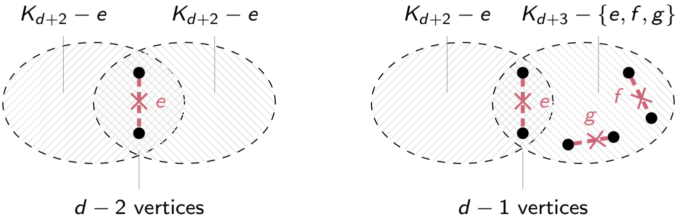
Flexible circuits

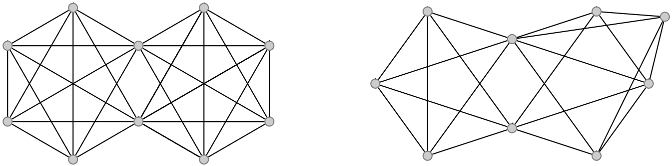
- It is easy to show that a vertex in an $\mathcal{R}_d$-circuit has degree at least $d+1$.
- K_{d+2} is the smallest $\mathcal{R}_d$-circuit.
- The smallest flexible $\mathcal{R}_d$-circuit in $\mathbb{R}^d$ is the double banana $B_{d,d-1}$:





Families of flexible circuits





- $B_{d,d-2}$ (which is defined for all $d \geq 4$) and the family $\mathcal{B}^{+}_{d,d-1}$.

Related work

- Tay 1993 - examples of flexible $\mathcal{R}_3$-circuits. Notably he gave examples of 4-connected flexible $\mathcal{R}_3$-circuits.
- Cheng, Sitharam and Streinu 2013 - construction of large flexible $\mathcal{R}_3$-circuits.

Theorem - GGJN

Let $G=(V,E)$ be a graph with $|V|\leq d+6$. Then G is $\mathcal{R}_d$-rigid if and only if G contains a spanning subgraph H that is $(d,\binom{d+1}{2})$-tight, d-connected and does not contain $B_{d,d-1}$ or $B_{d,d-2}$ as a subgraph.

- A very recent preprint of Jordán gives the same result when $|V|\leq d+4$ (with a different, simpler, proof). In particular his result implies that all $\mathcal{R}_d$-circuits on at most $d+4$ vertices are rigid.

Flexible circuits

Theorem - GGJN

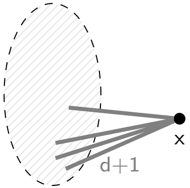
Let $G = (V, E)$ be a flexible $\mathcal{R}_d$-circuit with $|V| \leq d + 6$. Then either:
(a) $d = 3$ and $G \in \{B_{3,2}, B_{3,2}^+\}$ or
(b) $d \geq 4$ and $G \in \{B_{d,d-1}, B_{d,d-2}\} \cup \mathcal{B}_{d,d-1}^+$.

Proof 1

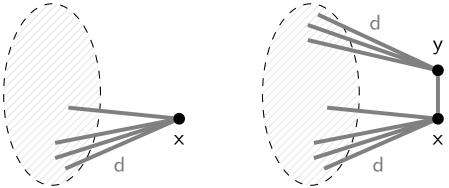
- Let $G = (V, E)$ be a counterexample to the theorem such that the dimension d is as small as possible, and subject to this condition, $|V|$ is as small as possible.
- G is a flexible $\mathcal{R}_d$-circuit so it is $(d, \binom{d+1}{2})$-sparse.
- Case 1: $\delta(G) = d + 1$. For any v with $d(v) = d + 1$ there are two non-adjacent neighbours x, y.
- Let $H = G - v + xy$. If H is $\mathcal{R}_d$-independent then so is G. Hence H contains a $\mathcal{R}_d$-circuit.
- The choice of G now implies that there is a rigid subgraph G' (containing x, y) of G. G' has at least $d + 2$ vertices and we analyse the options for the remaining (at most) 4 vertices.

Proof 2

- Let $X = V(G) - V(G')$. If $|X| = 1$ then G is rigid by 0-extension, a contradiction.

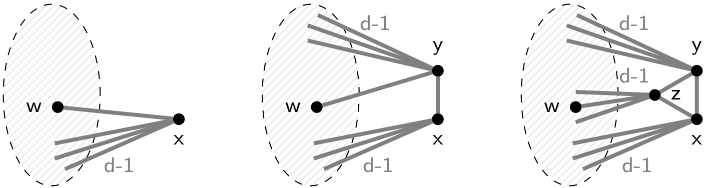


- If $|X| = 2$ then the same argument works.



Proof 3

- This argument also works if $|X| \in \{3, 4\}$ and $G[X]$ does not contain a spanning cycle.
- The remaining cases are harder and I'll just illustrate the easiest:

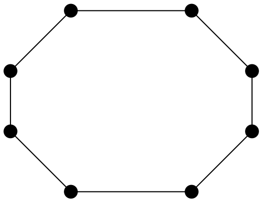


- When X has at least d neighbours in G', this sequence of extensions shows G is $\mathcal{R}_d$-rigid, a contradiction. When X has less neighbours in G' then we find our stated flexible circuits.

Proof 4

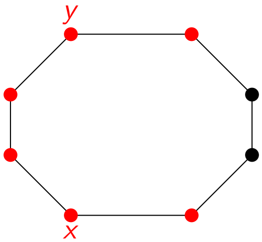
- Case 2: $\delta(G) \geq d+2$. We consider a vertex v with $d(v) = \Delta(G)$. If $d(v) = |V| - 1$, or $|V| - 2$, then we obtain a contradiction by coning.
- (Note we are finished if $|V| \leq d+4$.)
- If $|V| = d+5$, all that remains is that G is $(d+2)$-regular and hence $\overline{G}$ is 2-regular.
- Claim. There is some x, y to which we may apply vertex splitting to G/xy (with this claim we are done: G/xy is $(d, \binom{d+1}{2})$-sparse and hence $\mathcal{R}_d$-independent by the minimality of G, then apply Whiteley's vertex splitting result to show G is $\mathcal{R}_d$-independent).

Proof 5



- Since $d \geq 3$, $|V| \geq 8$ and hence it is easy to find two non-adjacent vertices $x, y \in \overline{G}$ with no common neighbours.
- In G, xy is an edge and x, y are in exactly $d-1$ $(= d+5-6)$ triangles.

Proof 5



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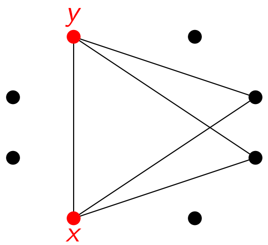

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Proof 6

- So $|V| = d + 6$ and $\overline{G}$ has $\delta(\overline{G}) \geq 2$ and $\Delta(\overline{G}) \leq 3$.
- If $\delta(\overline{G}) = 2$ and $\Delta(\overline{G}) = 3$ then we can use vertex splitting again.
- Therefore $\overline{G}$ is either 2-regular or 3-regular.
- In both cases we can determine that $|E| = d|V| - \binom{d+1}{2}$.
- Hence G is either 12-regular on 15 vertices (there are 17 such graphs) and $d = 9$ or G is 6-regular on 10 vertices (there are 21 such graphs) and $d = 4$.
- These can be checked to be $\mathcal{R}_d$-independent by computer, contradicting G being a $\mathcal{R}_d$-circuit and completing the proof.

Further work 1

- $|V| = d + 7$ seems plausible but there are technical difficulties in adapting our techniques. Going beyond $d + 7$ opens up more complicated types of $\mathcal{R}_d$-circuit.
- Graver, Servatius, Servatius - $K_{6,6}$ is a flexible $\mathcal{R}_4$-circuit.
- The iterated cone of $K_{6,6}$ is a $(d + 2)$-connected flexible $\mathcal{R}_d$-circuit on $d + 8$ vertices, for all $d \geq 4$.
- In general, what properties do flexible $\mathcal{R}_d$-circuits have?

Further work 2

- Let $G = (V, E)$, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. We say that G is a *t-sum of G_1, G_2 along an edge e* if $G = (G_1 \cup G_2) - e$, $G_1 \cap G_2 = K_t$ and $e \in E_1 \cap E_2$.
- Let G be a t-sum of G_1, G_2 for some $2 \leq t \leq d + 1$. We conjecture that G is an $\mathcal{R}_d$-circuit if and only if G_1, G_2 are $\mathcal{R}_d$-circuits.
- We can prove the case when $t = 2$ (for $d \leq 3$ this was already known) and give partial results in the general case.

Further work 3

- Jordán 2020 characterises global rigidity up to $|V| \leq d + 4$ vertices, by showing that the Hendrickson conditions are sufficient for such graphs. It would be natural to try and extend this.
- $|V| = d + 7$ may be difficult since $K_{5,5}$, when $d = 3$, is a problem. Connelly - $K_{5,5}$ satisfies Hendrickson's conditions but is not globally rigid in $\mathbb{R}^3$.
- Are Hendrickson's conditions sufficient for all $|V| \leq d + 6$?

Adverts

- Circle packings and geometric rigidity workshop, July 6-10, `https://icerm.brown.edu/topical_workshops/tw-20-cpgr/`

- Thematic program - geometric constraint systems, framework rigidity and distance geometry, January - June 2021, `http://www.fields.utoronto.ca/activities/20-21/constraint`

Thank you