Surface graphs, gain sparsity and some applications in discrete geometry

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Contact systems of line segments

What graphs arise as the intersection graphs of packings of line segments in the plane?

Definition 1

A 2-contact system is a finite collection of line segments in the plane having pairwise disjoint interiors and pairwise disjoint endpoints.

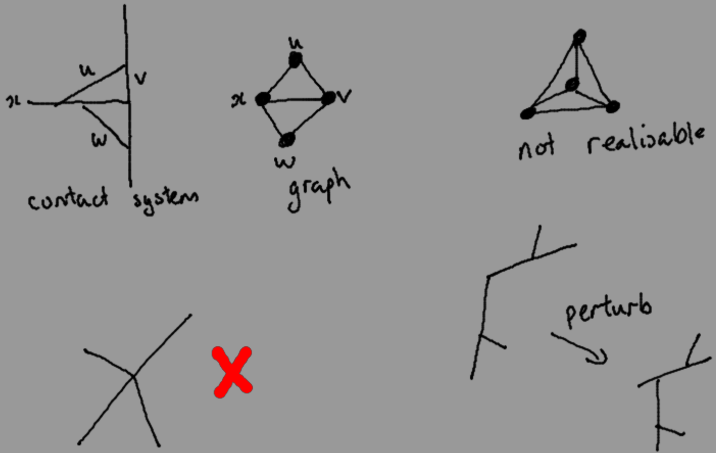
Hliněný ([4]): disjoint interiors and no point belongs to three segments. This is only (very) slightly more general and makes no difference for our investigations.

Theorem 2 (Thomassen [7])

A graph is the intersection graph of a finite 2-contact system of line segments in the plane if and only if it is planar and $(2,3)$-sparse.

Recall that graph (V,E) is $(2,3)$-sparse if $|E'| \leq 2|V(E')| - 3$ for all nonempty $E' \subset E$. This is equivalent to being a subgraph of a Laman graph.

Some diagrams

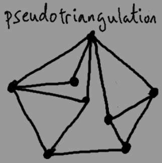


Comments

- A 2-contact representation (per Hliněný) can be perturbed to a 2-contact system without changing the intersection graph.
- Theorem 2 guarantees a polynomial time algorithm for the recognition problem for the class of graphs realisable by (finite) 2-contact systems.
- Variations and related problems:
  - packings of Jordan arcs, string graphs
  - k-contact systems for $k \geq 3$
  - polygon packings, circular arcs, wedges, ...
  - orthogonal collections of line segments
- C, Kitson, Power and Shakir: symmetric packings of circular arcs in the plane (see [1]).

Pseudotriangulations

Suppose G is a plane connected graph with straight line edges. A face F of G is a pseudotriangle if exactly three of its internal angles are $< \pi$. G is a pointed pseudotriangulation if every bounded face is a pseudotriangle, every vertex is pointed, and the unbounded face is the complement of the convex hull of $V(G)$.



Theorem 3 (Streinu [6], Haas et al. [3])

A graph has an embedding as a pointed pseudotriangulation if and only if it is a planar Laman graph.

Contributions

We consider symmetric versions of Theorems 2 and 3. A natural class of symmetry groups to consider are discrete subgroups of the Euclidean group.

- point groups (no translations)
- frieze groups (one independent translation)
- wallpaper groups (two independent translations)

Our results:

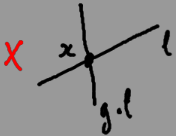
1. analogues of the theorems above for cyclic groups whose generator is either a rotation or a translation.
2. some partial results in other cases.
3. inductive characterisations for some interesting classes of topological graphs. These might interesting in other geometric/combinatorial contexts.

Symmetric contact systems

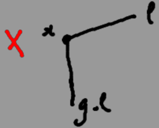
Definition 4

Let Γ be a discrete subgroup of the Euclidean group of isometries of the plane. A Γ-symmetric contact system of line segments is a 2-contact system $\mathcal{L}$ such that $g.l \in \mathcal{L}$ for all $g \in \Gamma, l \in L$, and $\mathcal{L}$ has finitely many Γ-orbits.

Let $\tilde{\Sigma}$ be the union of the free orbits of Γ. We can assume that $l \subset \tilde{\Sigma}$ for all $l \in \mathcal{L}$ without any significant loss of generality.

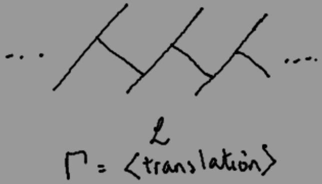


Suppose $x \in l$ is fixed by some $g \neq 1_\Gamma$. Then $x \in l \cap g.l$. It follows that $g.l = l$ and $l \cap m = \emptyset$ for $m \in \mathcal{L}, m \neq l$.

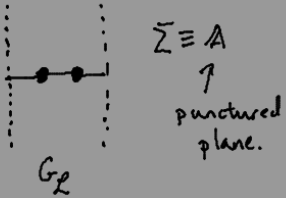


$\Sigma = \tilde{\Sigma}/\Gamma$ is the space of non-singular points of the orbifold $\mathbb{R}^2/\Gamma$: it is a smooth surface without boundary.

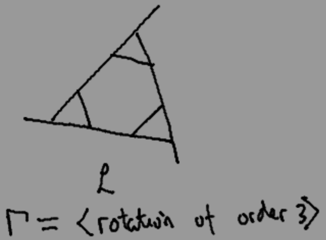
Examples



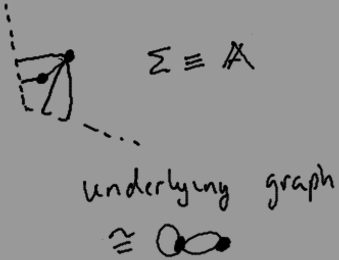
ℒ
Γ = ⟨translation⟩



G_ℒ
Σ ≡ A
punctured plane.



ℒ
Γ = ⟨rotation of order 3⟩



Σ ≡ A
underlying graph
≅

The graph of a symmetric contact system

Let $\mathcal{L}$ be a Γ-symmetric contact system and let D be the intersection graph of $\mathcal{L}$. Observe that

- Γ acts by graph automorphisms on D.
- D comes equipped with a natural plane embedding $\tilde{\Phi} : |D| \to \mathbb{R}^2$.
- We can choose $\tilde{\Phi}$ so that $\tilde{\Phi}(|D|) \subset \tilde{\Sigma}$ and so that $\tilde{\Phi}$ is Γ-equivariant.

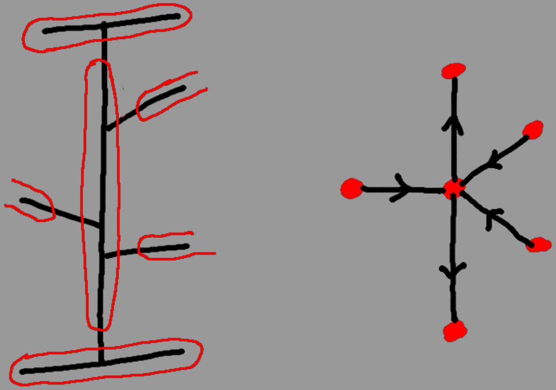
Thus we obtain $\Phi : |D/\Gamma| \to \Sigma$. Let

$$G_{\mathcal{L}} = (D/\Gamma, \Phi).$$

The topological graph $G_{\mathcal{L}}$ is the object that describes the combinatorics of the symmetric contact system $\mathcal{L}$.

Note that D/Γ may have loop edges and/or parallel edges.

Embedding the intersection graph



Is there an analogue of Theorem 2 for symmetric contact systems? The genus of Σ plays an important role.

Suppose that Σ is homeomorphic to

$$\mathbb{A} := \mathbb{R}^2 - \{(0,0)\}$$

Let G be an $\mathbb{A}$-graph. We say that G is **balanced** if some face of G contains both ends of $\mathbb{A}$, and **unbalanced** otherwise.

Define $f(G) = 2|V(G)| - |E(G)|$.

Definition 5

Let $l = 1, 2$. An $\mathbb{A}$-graph G is $(2,3,l)$-**sparse** if

- $f(H) \geq l$ for every subgraph H of G.
- $f(H) \geq 3$ for every balanced subgraph H of G that contains at least one edge.

If in addition, either $f(G) = l$, or G is balanced and $f(G) = 3$, or G is an isolated vertex, then we say that G is $(2,3,l)$-tight.

Results: symmetric contact systems

Theorem 6 (C, S)

Suppose that Γ is generated by a translation or by a rotation of order 2. An $\mathbb{A}$-graph G is the graph of a Γ-symmetric contact system if and only if G is $(2,3,2)$-sparse.

Theorem 7 (C, S)

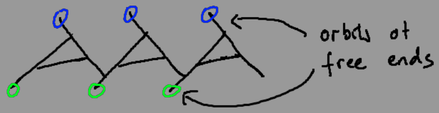
Suppose that Γ is generated by a a rotation of order at least 3. An $\mathbb{A}$-graph G is the graph of a Γ-symmetric contact system if and only if G is $(2,3,1)$-sparse.

These sparsity conditions are examples of gain sparsity conditions which are defined on gain graphs.

Proofs: necessity

Suppose Γ is generated by the translation $(x, y) \mapsto (x + 1, y)$. The other cases are similar.

- Given a Γ-symmetric contact system $\mathcal{L}$, then $f(G_{\mathcal{L}})$ counts the number of orbits of "free ends" in $\mathcal{L}$.
- For $H \leq G_{\mathcal{L}}$ you can find one free end by moving up through the segments in $V(H)$ and another by moving down. Thus $f(H) \geq 2$.
- If H is balanced then you can find a third free end by moving through segments in $V(H)$ perpendicular to the line joining the first two free ends.

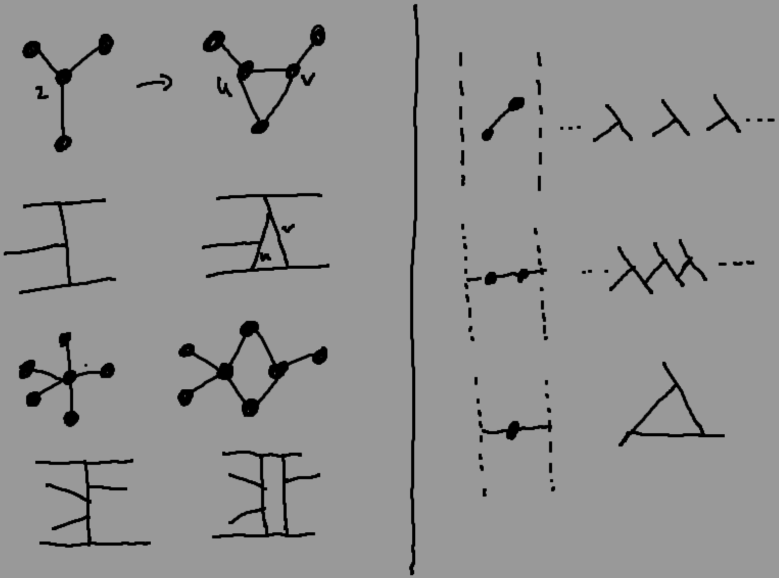


Proofs: sufficiency

Outline:

1. Let G be a $(2, 3, 2)$-sparse $\mathbb{A}$-graph. Then G can be completed to a $(2, 3, 2)$-tight $\mathbb{A}$-graph by adding edges only (not obvious!). On the other hand edge deletions are clearly realisable by contact systems.
2. If G is a $(2, 3, 2)$-tight $\mathbb{A}$-graph then there is a sequence of $(2, 3, 2)$-tight A-graphs $G_1, \cdots, G_n = G$ where G_{i+1} is obtained from G_i by a triangle vertex split or a quadrilateral vertex split and G_1 has two vertices.
3. The base graphs are realisable by symmetric contact systems of line segments and the splitting moves are also realisable.

Splitting moves and base graph realisations



Some related results in the literature:

- Fekete, Jordán and Whiteley have an inductive characterisation of plane Laman graphs using triangle splits only ([2])
- There are related inductive characterisations of $(2, 2)$-tight $\mathbb{A}$-graphs and torus graphs in [1].
- Inductive characterisations of quadrangulations (and triangulations) of various surfaces have been investigated by various people (for example [5])

Symmetric pseudotriangulations

Observe that the splitting moves are also realisable by symmetric pseudotriangulations.

Definition 8

A Γ-symmetric pointed pseudotriangulation is a plane graph P with straight line edges such that

1. P is Γ-invariant.
2. every vertex is pointed.
3. every cellular face (i.e bounded and containing no singular points) is a pseudotriangle and the number of convex angles in a non-cellular face is minimal.

The quotient graph P/Γ is a Σ-graph[1] since $P \subset \tilde{\Sigma}$

[1] there is one exception that can arise when Γ has a rotation of order two

Theorem 9 (C, S)

Let $\Gamma = \langle \sigma \rangle$ where σ is a translation or a rotation of order two. An $\mathbb{A}$-graph G is isomorphic to P/Γ for some Γ-symmetric pointed pseudotriangulation P if and only if G is $(2,3,2)$-tight.

Theorem 10 (C, S)

Let $\Gamma = \langle \sigma \rangle$ where σ is a rotation of order at least three. An $\mathbb{A}$-graph G is isomorphic to P/Γ for some Γ-symmetric pointed pseudotriangulation P if and only if G is $(2,3,1)$-tight.

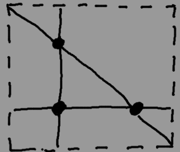
Positive genus

When Σ has positive genus things can be trickier.

Let $\Gamma = \langle \sigma, \tau \rangle$ where σ, τ are independent translations. In this case Σ is a torus.

Theorem 11

The graph of a Γ-symmetric contact system of line segments is a $(2, 3, 2, 0)$-tight torus graph.



In this case the inductive construction must have relatively large base graphs.

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